

A GEOMETRIC CONSTRUCTION FOR INVARIANT JET DIFFERENTIALS

GERGELY BERCZI AND FRANCES KIRWAN
MATHEMATICAL INSTITUTE, OXFORD OX1 3BJ, UK

1. INTRODUCTION

The action of the reparametrization group \mathbb{G}_k , consisting of k -jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, on the bundle $J_k = J_k T^*X$ of k -jets at 0 of germs of holomorphic curves $f : \mathbb{C} \rightarrow X$ in a complex manifold X has been a focus of investigation since the work of Demailly [5] which built on that of Green and Griffiths [13]. Here \mathbb{G}_k is a non-reductive complex algebraic group which is the semi-direct product $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ of its unipotent radical \mathbb{U}_k with \mathbb{C}^* ; it has the form

$$\mathbb{G}_k \cong \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & \cdots & & \\ 0 & 0 & \alpha_1^3 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \dots, \alpha_k \in \mathbb{C} \right\}$$

where the entries above the leading diagonal are polynomials in $\alpha_1, \dots, \alpha_k$, and \mathbb{U}_k is the subgroup consisting of matrices of this form with $\alpha_1 = 1$. The bundle of Demailly-Semple jet differentials of order k over X has fibre at $x \in X$ given by the algebra $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ of \mathbb{U}_k -invariant polynomial functions on the fibre $(J_k)_x = (J_k T^*X)_x$ of $J_k T^*X$. More generally following [25] we can replace \mathbb{C} with \mathbb{C}^p for $p \geq 1$ and consider the bundle $J_{k,p} T^*X$ of k -jets at 0 of holomorphic maps $f : \mathbb{C}^p \rightarrow X$ and the reparametrization group $\mathbb{G}_{k,p}$ consisting of k -jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$; then $\mathbb{G}_{k,p}$ is the semi-direct product of its unipotent radical $\mathbb{U}_{k,p}$ and the complex reductive group $\mathrm{GL}(p)$, while its subgroup $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes \mathrm{SL}(p)$ (which equals $\mathbb{U}_{k,p}$ when $p = 1$) fits into an exact sequence $1 \rightarrow \mathbb{G}'_{k,p} \rightarrow \mathbb{G}_{k,p} \rightarrow \mathbb{C}^* \rightarrow 1$. The generalized Demailly-Semple algebra is then $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

The Demailly-Semple algebras $\mathcal{O}(J_k)^{\mathbb{U}_k}$ and their generalizations have been studied for a long time. The invariant jet differentials play a crucial role in the strategy devised by Green, Griffiths [13], Bloch [4], Demailly [5, 6], Siu [28, 29, 30] and others to prove Kobayashi's 1970 hyperbolicity conjecture [19] and the related conjecture of Green and Griffiths in the special case of hypersurfaces in projective space. This strategy has been recently used successfully by Diverio, Merker and Rousseau in [7] and then by the first

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author in [1] to give effective lower bounds for the degrees of generic hypersurfaces in \mathbb{P}_n for which the Green-Griffiths conjecture holds.

In particular it has been a long-standing problem to determine whether the algebras of invariants $O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ and bi-invariants $O((J_{k,p})_x)^{\mathbb{G}'_{k,p} \times U_{n,x}}$ (where $U_{n,x}$ is a maximal unipotent subgroup of $GL(T_x X) \cong GL(n)$) are finitely generated as graded complex algebras, and if so to provide explicit finite generating sets. In [20] Merker showed that when $p = 1$ and both k and $n = \dim X$ are small then these algebras are finitely generated, and for $p = 1$ and all k and n he provided an algorithm which produces finite sets of generators when they exist. In this paper we will describe methods inspired by [2] and the approach of [9] to non-reductive geometric invariant theory (GIT) to prove the finite generation of $O((J_k)_x)^{\mathbb{U}_k}$ for all n and $k \geq 2$ (from which the finite generation of the corresponding bi-invariants follows). In fact we will show that \mathbb{U}_k is a Grosshans subgroup of $SL(k)$, so that the algebra $O(SL(k))^{\mathbb{U}_k}$ is finitely generated and hence every linear action of \mathbb{U}_k which extends to a linear action of $SL(k)$ has finitely generated invariants. We will also give a geometric description of a finite set of generators for $O(SL(k))^{\mathbb{U}_k}$, and a geometric description of the associated affine variety

$$SL(k)//\mathbb{U}_k = \text{Spec}(O(SL(k))^{\mathbb{U}_k})$$

which leads to a geometric description of the affine variety

$$(J_k)_x//\mathbb{U}_k = \text{Spec}(O((J_k)_x)^{\mathbb{U}_k})$$

as a GIT quotient

$$((J_k)_x \times (SL(k)//\mathbb{U}_k))/SL(k)$$

by the reductive group $SL(k)$, in the sense of classical geometric invariant theory [23]. Similarly we expect that if $p > 1$ and k is sufficiently large (depending on p) then $\mathbb{G}'_{k,p}$ is a subgroup of $SL(\text{sym}^{\leq k}(p))$, where

$$\text{sym}^{\leq k}(p) = \sum_{i=1}^k \dim \text{Sym}^i \mathbb{C}^p,$$

such that the algebra $O(SL(\text{sym}^{\leq k}(p)))^{\mathbb{G}'_{k,p}}$ is finitely generated, and thus that the algebra and $O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ is also finitely generated, and we have a geometric description of the associated affine variety

$$(J_{k,p})_x//\mathbb{G}'_{k,p}.$$

The layout of this paper is as follows. §2 reviews the reparametrization groups \mathbb{G}_k and $\mathbb{G}_{k,p}$ and their actions on jet bundles and jet differentials over a complex manifold X . Next §3 reviews some of the results of [9] on non-reductive geometric invariant theory. In §4 we recall from [2] a geometric description of the quotients by \mathbb{U}_k and \mathbb{G}_k of open subsets of $(J_k)_x$, and in §5 this is used to find explicit affine and projective embeddings of these quotients and explicit embeddings of $SL(k)/\mathbb{U}_k$. In §6 we see that the complement of $SL(k)/\mathbb{U}_k$ in its closure for a suitable embedding in an affine space has codimension at least two. In §7 we conclude that \mathbb{U}_k is a Grosshans subgroup of

$SL(k)$ when $k \geq 2$, so that $O(SL(k))^{\mathbb{U}_k}$ and $O((J_k)_x)^{\mathbb{U}_k}$ are finitely generated, and provide a geometric description of a finite set of generators of $O(SL(k))^{\mathbb{U}_k}$. Finally §8 and §9 discuss how to extend the results of §6 and §7 to the action of $\mathbb{G}_{k,p}$ on the jet bundle $J_{k,p} \rightarrow X$ of k -jets of germs of holomorphic maps from \mathbb{C}^p to X for $p > 1$.

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2. JETS OF CURVES AND JET DIFFERENTIALS

Let X be a complex n -dimensional manifold and let k be a positive integer. Green and Griffiths in [13] introduced the bundle $J_k \rightarrow X$ of k -jets of germs of parametrized curves in X ; its fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \rightarrow (X, x)$, with the equivalence relation $f \sim g$ if and only if the derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates (z_1, \dots, z_n) on an open neighbourhood $\Omega \subset X$ around x , the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \dots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of \mathbb{C}^n -valued maps

$$f = (f_1, f_2, \dots, f_n)$$

on open neighbourhoods of 0 in \mathbb{C} . Thus in these coordinates the fibre is

$$J_{k,x} = \{(f'(0), \dots, f^{(k)}(0)/k!)\} = (\mathbb{C}^n)^k,$$

which we identify with \mathbb{C}^{nk} . Note, however, that J_k is not a vector bundle over X , since the transition functions are polynomial, but not linear.

Let \mathbb{G}_k be the group of k -jets at the origin of local reparametrizations of $(\mathbb{C}, 0)$

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \dots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_k by substitution. A short computation shows that this is a linear action on the fibre:

$$\begin{aligned} f \circ \varphi(t) &= f'(0) \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k) + \frac{f''(0)}{2!} \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k)^2 + \dots \\ &\quad \dots + \frac{f^{(k)}(0)}{k!} \cdot (\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_k t^k)^k \pmod{t^{k+1}} \end{aligned}$$

so the linear action of φ on the k -jet $(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$(1) \quad (f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1} + \dots + \alpha_{k-1}\alpha_1 \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2} + \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix}$$

where the matrix has general entry

$$(G_k)_{i,j} = \sum_{s_1 \geq 1, \dots, s_i \geq 1, s_1 + \dots + s_i = j} \alpha_{s_1} \dots \alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$(2) \quad 1 \rightarrow \mathbb{U}_k \rightarrow \mathbb{G}_k \rightarrow \mathbb{C}^* \rightarrow 1,$$

where $\mathbb{G}_k \rightarrow \mathbb{C}^*$ is the morphism $\varphi \rightarrow \varphi'(0) = \alpha_1$ in the notation used above, and

$$\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$$

is a semi-direct product. With the above identification, \mathbb{C}^* is the subgroup of \mathbb{G}_k consisting of diagonal matrices satisfying $\alpha_2 = \dots = \alpha_k = 0$ and \mathbb{U}_k is the unipotent radical of \mathbb{G}_k , consisting of matrices of the form above with $\alpha_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on k -jets is thus described by

$$\lambda \cdot (f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) = (\lambda f'(0), \lambda^2 f''(0)/2!, \dots, \lambda^k f^{(k)}(0)/k!)$$

Let $\mathcal{E}_{k,m}^n$ denote the vector space of complex valued polynomial functions

$$Q(u_1, u_2, \dots, u_k)$$

of $u_1 = (u_{1,1}, \dots, u_{1,n}), \dots, u_k = (u_{k,1}, \dots, u_{k,n})$ of weighted degree m with respect to this \mathbb{C}^* action, where $u_i = f^{(i)}(0)/i!$; that is, such that

$$Q(\lambda u_1, \lambda^2 u_2, \dots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \dots, u_k).$$

Thus elements of $\mathcal{E}_{k,m}^n$ have the form

$$Q(u_1, u_2, \dots, u_k) = \sum_{|i_1| + 2|i_2| + \dots + k|i_k| = m} u_1^{i_1} u_2^{i_2} \dots u_k^{i_k},$$

where $i_1 = (i_{1,1}, \dots, i_{1,n}), \dots, i_k = (i_{k,1}, \dots, i_{k,n})$ are multi-indices of length n . There is an induced action of \mathbb{G}_k on the algebra $\bigoplus_{m \geq 0} \mathcal{E}_{k,m}^n$. Following Demailly (see [5]), we denote by $E_{k,m}^n$ (or $E_{k,m}$) the Demailly-Semple bundle whose fibre at x consists of the \mathbb{U}_k -invariant polynomials on the fibre of J_k at x of weighted degree m , i.e those which satisfy

$$\begin{aligned} & Q((f \circ \varphi)'(0), (f \circ \varphi)''(0)/2!, \dots, (f \circ \varphi)^{(k)}(0)/k!) \\ &= \varphi'(0)^m \cdot Q(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!), \end{aligned}$$

and we let $E_k^n = \oplus_m E_{k,m}^n$ denote the Demailly-Semple bundle of graded algebras of invariants.

We can also consider higher dimensional holomorphic surfaces in X , and therefore we fix a parameter $1 \leq p \leq n$, and study germs of maps $\mathbb{C}^p \rightarrow X$.

Again we fix the degree k of our map, and introduce the bundle $J_{k,p} \rightarrow X$ of k -jets of maps $\mathbb{C}^p \rightarrow X$. The fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}^p, 0) \rightarrow (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$.

We need a description of the fibre $J_{k,p,x}$ in terms of local coordinates as in the case when $p = 1$. Let (z_1, \dots, z_n) be local holomorphic coordinates on an open neighbourhood $\Omega \subset X$ around x , and let (u_1, \dots, u_p) be local coordinates on \mathbb{C}^p . The elements of the fibre $J_{k,p,x}$ are \mathbb{C}^n -valued maps

$$f = (f_1, f_2, \dots, f_n)$$

on \mathbb{C}^p , and two maps represent the same jet if their Taylor expansions around $\mathbf{z} = 0$

$$f(\mathbf{z}) = x + \mathbf{z}f'(0) + \frac{\mathbf{z}^2}{2!}f''(0) + \dots + \frac{\mathbf{z}^k}{k!}f^{(k)}(0) + O(\mathbf{z}^{k+1})$$

coincide up to order k . Note that here

$$f^{(i)}(0) \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^n)$$

and in these coordinates the fibre is a finite-dimensional vector space

$$J_{k,p,x} = \{(f'(0), \dots, f^{(k)}(0)/k!)\} \cong \mathbb{C}^{n \binom{k+p-1}{k-1}}.$$

Let $\mathbb{G}_{k,p}$ be the group of k -jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$. Elements of $\mathbb{G}_{k,p}$ are represented by holomorphic maps

$$(3) \quad \mathbf{u} \rightarrow \varphi(\mathbf{u}) = \Phi_1 \mathbf{u} + \Phi_2 \mathbf{u}^2 + \dots + \Phi_k \mathbf{u}^k = \sum_{\mathbf{i} \in \mathbb{Z}^p \setminus \{0\}} a_{i_1 \dots i_p} u_1^{i_1} \dots u_p^{i_p}, \quad \Phi_1 \text{ is non-degenerate}$$

where $\Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$. The group $\mathbb{G}_{k,p}$ admits a natural fibrewise right action on $J_{k,p}$, by reparametrizing the k -jets of holomorphic p -discs. A computation similar to that in [2] shows that

$$f \circ \varphi(\mathbf{u}) = f'(0)\Phi_1 \mathbf{u} + (f'(0)\Phi_2 + \frac{f''(0)}{2!}\Phi_1^2)\mathbf{u}^2 + \dots + \sum_{i_1 + \dots + i_l = d} \frac{f^{(l)}(0)}{l!}\Phi_{i_1} \dots \Phi_{i_l} \mathbf{u}^l.$$

This defines a linear action of $\mathbb{G}_{k,p}$ on the fibres $J_{k,p,x}$ of $J_{k,p}$ with the matrix representation given by

$$(4) \quad \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \dots & \\ 0 & 0 & \Phi_1^3 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & \Phi_1^k \end{pmatrix},$$

where

- $\Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$ is a $p \times \dim(\text{Sym}^i \mathbb{C}^p)$ -matrix, the i th degree component of the map Φ , which is represented by a map $(\mathbb{C}^p)^{\otimes i} \rightarrow \mathbb{C}^p$;
- $\Phi_{i_1} \dots \Phi_{i_l}$ is the matrix of the map $\text{Sym}^{i_1+\dots+i_l}(\mathbb{C}^p) \rightarrow \text{Sym}^l \mathbb{C}^p$, which is represented by

$$\sum_{\sigma \in S_l} \Phi_{i_1} \otimes \dots \otimes \Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1} \otimes \dots \otimes (\mathbb{C}^p)^{\otimes i_l} \rightarrow (\mathbb{C}^p)^{\otimes l};$$

- the (l, m) block of $\mathbb{G}_{k,p}$ is $\sum_{i_1+\dots+i_l=m} \Phi_{i_1} \dots \Phi_{i_l}$. The entries in these boxes are indexed by pairs (τ, μ) where $\tau \in \binom{p+l-1}{l-1}, \mu \in \binom{p+m-1}{m-1}$ correspond to bases of $\text{Sym}^l(\mathbb{C}^p)$ and $\text{Sym}^m(\mathbb{C}^p)$.

Example 2.1. For $p = 2, k = 3$, using the standard basis

$$\{e_i, e_i e_j, e_i e_j e_k : 1 \leq i \leq j \leq k \leq 2\}$$

of $(J_{3,2})_x$, we get the following 9×9 matrix for a general element of $\mathbb{G}_{3,2}$:

$$(5) \quad \begin{pmatrix} \alpha_{10} & \alpha_{01} & \alpha_{20} & \alpha_{11} & \alpha_{02} & \alpha_{30} & \alpha_{21} & \alpha_{12} & \alpha_{03} \\ \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} & \beta_{21} & \beta_{12} & \beta_{03} \\ 0 & 0 & \alpha_{10}^2 & \alpha_{10}\alpha_{01} & \alpha_{01}^2 & \alpha_{10}\alpha_{20} & \alpha_{10}\alpha_{11} + \alpha_{01}\alpha_{20} & \alpha_{10}\alpha_{02} + \alpha_{11}\alpha_{01} & \alpha_{01}\alpha_{02} \\ 0 & 0 & \alpha_{10}\beta_{10} & \alpha_{10}\beta_{01} + \alpha_{01}\beta_{10} & \alpha_{01}\beta_{01} & \alpha_{10}\beta_{20} + \alpha_{20}\beta_{10} & P & Q & \alpha_{01}\beta_{02} + \alpha_{02}\beta_{01} \\ 0 & 0 & \beta_{10}^2 & \beta_{10}\beta_{01} & \beta_{01}^2 & \beta_{10}\beta_{20} & \beta_{10}\beta_{11} + \beta_{20}\beta_{01} & \beta_{01}\beta_{11} + \beta_{02}\beta_{10} & \beta_{01}\beta_{02} \\ 0 & 0 & 0 & 0 & 0 & \alpha_{10}^3 & \alpha_{10}^2\alpha_{01} & \alpha_{10}\alpha_{01}^2 & \alpha_{01}^3 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{10}^2\beta_{10} & \alpha_{10}\alpha_{10}\beta_{01} & \alpha_{10}\alpha_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{10}\beta_{10}^2 & \alpha_{10}\beta_{10}\beta_{01} & \alpha_{10}\beta_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\ 0 & 0 & 0 & 0 & 0 & \beta_{10}^3 & \beta_{10}^2\beta_{01} & \beta_{10}\beta_{01}^2 & \beta_{01}^3 \end{pmatrix}$$

where

$$P = \alpha_{10}\beta_{11} + \alpha_{11}\beta_{10} + \alpha_{20}\beta_{01} + \alpha_{01}\beta_{20} \text{ and } Q = \alpha_{01}\beta_{11} + \alpha_{11}\beta_{01} + \alpha_{02}\beta_{10} + \alpha_{10}\beta_{02}.$$

This is a subgroup of the standard parabolic $P_{2,3,4} \subset GL(9)$. The diagonal blocks are the representations $\text{Sym}^i \mathbb{C}^2$ for $i = 1, 2, 3$ of $GL(2)$, where \mathbb{C}^2 is the standard representation of $GL(2)$.

In general the linear group $\mathbb{G}_{k,p}$ is generated along its first p rows; that is, the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The assumption on the parameters is that the determinant of the

smallest diagonal $p \times p$ block is nonzero; for the $p = 2, k = 3$ example above this means that

$$\det \begin{pmatrix} \alpha_{10} & \alpha_{01} \\ \beta_{10} & \beta_{01} \end{pmatrix} \neq 0.$$

The parameters in the $(1, m)$ block are indexed by a basis of $\text{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p$, so they are of the form α_ν^l , where $\nu \in \binom{p+m-1}{m-1}$ is an m -tuple and $1 \leq l \leq p$. An easy computation shows that:

Proposition 2.2. *The polynomial in the (l, m) block and entry indexed by*

$$\tau = (\tau[1], \dots, \tau[l]) \in \binom{p+l-1}{l-1}$$

and $\nu \in \binom{p+m-1}{m-1}$ is

$$(6) \quad (\mathbb{G}_{k,p})_{\tau,\nu} = \sum_{\nu_1 + \dots + \nu_l = \nu} \alpha_{\nu_1}^{\tau[1]} \alpha_{\nu_2}^{\tau[2]} \dots \alpha_{\nu_l}^{\tau[l]}$$

Note that $\mathbb{G}_{k,p}$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ by $GL(p)$; that is, we have an exact sequence

$$1 \rightarrow \mathbb{U}_{k,p} \rightarrow \mathbb{G}_{k,p} \rightarrow GL(p) \rightarrow 1,$$

and $\mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{U}_{k,p} \rtimes GL(p)$. Here $\mathbb{G}_{k,p}$ has dimension $p \times \text{sym}^{\leq k}(p)$ where $\text{sym}^{\leq k}(p) = \dim(\oplus_{i=1}^k \text{Sym}^i \mathbb{C}^p)$, and is a subgroup of the standard parabolic subgroup $P_{p, \text{sym}^2(p), \dots, \text{sym}^k(p)}$ of $GL(\text{sym}^{\leq k}(p))$ where $\text{sym}^i(p) = \dim(\text{Sym}^i \mathbb{C}^p)$. We define $\mathbb{G}'_{k,p}$ to be the subgroup of $\mathbb{G}_{k,p}$ which is the semi-direct product

$$\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes SL(p)$$

(so that $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p}$ when $p = 1$) fitting into the exact sequence

$$1 \rightarrow \mathbb{U}_{k,p} \rightarrow \mathbb{G}'_{k,p} \rightarrow SL(p) \rightarrow 1.$$

The action of the maximal torus $(\mathbb{C}^*)^p \subset GL(p)$ of the Levi subgroup of $\mathbb{G}_{k,p}$ is

$$(7) \quad (\lambda_1, \dots, \lambda_p) \cdot f^{(i)} = (\lambda_1^i \frac{\partial^i f}{\partial u_1^i}, \dots, \lambda_1^{i_1} \dots \lambda_p^{i_p} \frac{\partial^i f}{\partial u_1^{i_1} \dots \partial u_p^{i_p}} \dots \lambda_p^i \frac{\partial^i f}{\partial u_p^i})$$

We introduce the *Green-Griffiths* vector bundle $E_{k,p,m}^{GG} \rightarrow X$, whose fibres are complex-valued polynomials

$$Q(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$$

on the fibres of $J_{k,p}$, having weighted degree (m, \dots, m) with respect to the action (7) of $(\mathbb{C}^*)^p$. That is, for $Q \in E_{k,p,m}^{GG}$

$$Q(\lambda f'(0), \lambda f''(0)/2!, \dots, \lambda f^{(k)}(0)/k!) = \lambda_1^m \dots \lambda_p^m Q(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$$

for all $\lambda \in \mathbb{C}^p$ and $(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) \in J_{k,p,m}$.

Definition 2.3. *The generalized Demailly-Semple bundle $E_{k,p,m} \rightarrow X$ over X has fibre consisting of the $\mathbb{G}'_{k,p}$ -invariant jet differentials of order k and weighted degree (m, \dots, m) ; that is, the complex-valued polynomials $Q(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$ on the fibres of $J_{k,p}$ which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as*

$$Q(f \circ \phi) = (J_\phi)^m Q(f) \circ \phi,$$

where $J_\phi = \det \Phi_1$ denotes the Jacobian of ϕ at 0. The generalized Demailly-Semple bundle of algebras $E_{k,p} = \bigoplus_{m \geq 0} E_{k,p,m}$ is the associated graded algebra of $\mathbb{G}'_{k,p}$ -invariants, whose fibre at $x \in X$ is the generalized Demailly-Semple algebra $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

The determination of a suitable generating set for the invariant jet differentials when $p = 1$ is important in the longstanding strategy to prove the Green-Griffiths conjecture. It has been suggested in a series of papers [13, 5, 27, 20, 7, 21] that the Schur decomposition of the Demailly-Semple algebra, together with good estimates of the higher Betti numbers of the Schur bundles and an asymptotic estimation of the Euler characteristic, should result in a positive lower bound for the global sections of the Demailly-Semple jet differential bundle.

3. GEOMETRIC INVARIANT THEORY

Suppose now that Y is a complex quasi-projective variety on which a linear algebraic group G acts. For geometric invariant theory (GIT) we need a linearization of the action; that is, a line bundle L on Y and a lift \mathcal{L} of the action of G to L . Usually L is ample, and hence (as it makes no difference for GIT if we replace L with $L^{\otimes k}$ for any integer $k > 0$) we can assume that for some projective embedding $Y \subseteq \mathbb{P}^n$ the action of G on Y extends to an action on \mathbb{P}^n given by a representation $\rho : G \rightarrow GL(n+1)$, and take for L the hyperplane line bundle on \mathbb{P}^n .

For classical GIT developed by Mumford [23] (cf. also [8, 22, 24, 26]) we require the complex algebraic group G to be reductive. Let Y be a projective complex variety with an action of a complex reductive group G and linearization \mathcal{L} with respect to an ample line bundle L on Y . Then $y \in Y$ is *semistable* for this linear action if there exists some $m > 0$ and $f \in H^0(Y, L^{\otimes m})^G$ not vanishing at y , and y is *stable* if also the action of G on the open subset

$$Y_f := \{x \in Y \mid f(x) \neq 0\}$$

is closed with all stabilizers finite. Y^{ss} has a projective categorical quotient $Y^{ss} \rightarrow Y//G$, which restricts on the set of stable points to a geometric quotient $Y^s \rightarrow Y^s/G$ (see [23] Theorem 1.10). The morphism $Y^{ss} \rightarrow Y//G$ is surjective, and identifies $x, y \in Y^{ss}$ if and only if the closures of the G -orbits of x and y meet in Y^{ss} ; moreover each point in $Y//G$ is represented by a unique closed G -orbit in Y^{ss} . There is an induced action of G on the homogeneous coordinate ring

$$\hat{\mathcal{O}}_L(Y) = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})$$

of Y . The subring $\hat{\mathcal{O}}_L(Y)^G$ consisting of the elements of $\hat{\mathcal{O}}_L(Y)$ left invariant by G is a finitely generated graded complex algebra because G is reductive, and the GIT quotient $Y//G$ is the projective variety $\text{Proj}(\hat{\mathcal{O}}_L(Y)^G)$ [23]. The subsets Y^{ss} and Y^s of Y are characterized by the following properties (see [23, Chapter 2] or [24]).

Proposition 3.1. (*Hilbert-Mumford criteria*) (i) *A point $x \in Y$ is semistable (respectively stable) for the action of G on Y if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus of G .*

(ii) *A point $x \in Y$ with homogeneous coordinates $[x_0 : \dots : x_n]$ in some coordinate system on \mathbb{P}^n is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on \mathbb{P}^n with weights $\alpha_0, \dots, \alpha_n$ if and only if the convex hull*

$$\text{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 (respectively contains 0 in its interior).

Similarly if a complex reductive group G acts linearly on an affine variety Y then we have a GIT quotient

$$Y//G = \text{Spec}(O(Y)^G)$$

which is the affine variety associated to the finitely generated algebra $O(Y)^G$ of G -invariant regular functions on Y . In this case $Y^{ss} = Y$ and the inclusion $O(Y)^G \hookrightarrow O(Y)$ induces a morphism of affine varieties $Y \rightarrow Y//G$.

Now suppose that H is any complex linear algebraic group, with unipotent radical $U \triangleleft H$ (so that $R = H/U$ is reductive and H is isomorphic to the semi-direct product $U \rtimes R$), acting linearly on a complex projective variety Y with respect to an ample line bundle L . Then $\text{Proj}(\hat{\mathcal{O}}_L(Y)^H)$ is not in general well-defined as a projective variety, since the ring of invariants

$$\hat{\mathcal{O}}_L(Y)^H = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^H$$

is not necessarily finitely generated as a graded complex algebra, and so it is not obvious how GIT might be generalised to this situation (cf. [9, 11, 10, 14, 15, 18]). However in some cases it is known that $\hat{\mathcal{O}}_L(Y)^U$ is finitely generated, which implies that

$$\hat{\mathcal{O}}_L(Y)^H = \left(\bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^U \right)^{H/U}$$

is finitely generated and hence the *enveloping quotient* in the sense of [9] is given by the associated projective variety

$$Y//H = \text{Proj}(\hat{\mathcal{O}}_L(Y)^H).$$

Similarly if Y is affine and H acts linearly on Y with $O(Y)^H$ finitely generated, then we have the enveloping quotient

$$Y//H = \text{Spec}(O(Y)^H).$$

There is a morphism

$$q : Y^{ss} \rightarrow Y//H,$$

from an open subset Y^{ss} of Y (where $Y^{ss} = Y$ when Y is affine), which restricts to a geometric quotient

$$q : Y^s \rightarrow Y^s/H$$

for an open subset $Y^s \subset Y^{ss}$. However in contrast with the reductive case, the morphism $q : Y^{ss} \rightarrow Y//H$ is not in general surjective; indeed the image of q is not in general a subvariety of $Y//H$, but is only a constructible subset.

If there is a complex reductive group G containing the unipotent radical U of H such that the algebra $\mathcal{O}(G)^U$ is finitely generated and the action of U on Y extends to a linear action of G , then

$$\mathcal{O}(Y)^U \cong (\mathcal{O}(Y) \otimes \mathcal{O}(G)^U)^G$$

is finitely generated and hence so is

$$\mathcal{O}(Y)^H = (\mathcal{O}(Y)^U)^{H/U}$$

(or if Y is projective with an ample linearisation L then $\hat{\mathcal{O}}_L(Y)^U$ is finitely generated and hence so is $\hat{\mathcal{O}}_L(Y)^H$). In this situation we say that U is a Grosshans subgroup of G (cf. [16, 17]). Then geometrically G/U is a quasi-affine variety with $\mathcal{O}(G/U) \cong \mathcal{O}(G)^U$, and it has a canonical affine embedding as an open subvariety of the affine variety

$$G//U = \text{Spec}(\mathcal{O}(G)^U)$$

with complement of codimension at least two. Moreover if a linear action of U on an affine variety Y extends to a linear action of G then

$$Y//U \cong (Y \times G//U)//G$$

(and a corresponding result is true if Y is projective). Conversely if we can find an embedding of G/U as an open subvariety of an affine variety Z with complement of codimension at least two, then

$$\mathcal{O}(G)^U \cong \mathcal{O}(Z)$$

is finitely generated and $G//U \cong Z$.

Suppose that U is a unipotent group with a reductive group R of automorphisms of U given by a homomorphism $\phi : R \rightarrow \text{Aut}(U)$ such that R contains a central one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow R$ for which the weights of the induced \mathbb{C}^* action on the Lie algebra \mathfrak{u} of U are all nonzero. Then we can form the semi-direct product

$$\hat{U} = \mathbb{C}^* \ltimes U \subseteq R \ltimes U$$

given by $\mathbb{C}^* \times U$ with group multiplication

$$(z_1, u_1) \cdot (z_2, u_2) = (z_1 z_2, (\lambda(z_2^{-1})(u_1))u_2).$$

The groups $\mathbb{G}_k = \mathbb{U}_k \ltimes \mathbb{C}^*$ and $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \ltimes \text{GL}(p)$ which act on the fibres of the jet bundles J_k and $J_{k,p}$ are of this form. We will use this structure to study the Demailly-Semple algebras of invariant jet differentials E_k^n and $E_{k,p}^n$ and prove

Theorem 3.2. *The fibres $O((J_k)_x)^{\mathbb{U}_k}$ and $O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ of the bundles E_k^n and $E_{k,p}^n$ are finitely generated graded complex algebras.*

Thus we have non-reductive GIT quotients

$$(J_k)_x // \mathbb{U}_k = \text{Spec}(O((J_k)_x)^{\mathbb{U}_k})$$

and

$$(J_{k,p})_x // \mathbb{G}'_{k,p} = \text{Spec}(O((J_{k,p})_x)^{\mathbb{G}'_{k,p}})$$

and we would like to understand them geometrically. There is a crucial difference here from the case of reductive group actions, even though the invariants are finitely generated: when H is a non-reductive group we cannot describe $Y//H$ geometrically as Y^{ss} modulo some equivalence relation. Instead our aim is to use methods inspired by [2] to study these geometric invariant theoretic quotients and the associated algebras of invariants.

Here a crucial ingredient would be to find an open subset W of $(J_{k,p})_x$ with a geometric quotient $W/\mathbb{G}'_{k,p}$ embedded as an open subset of an affine variety Z such that the complement of $W/\mathbb{G}'_{k,p}$ in Z has (complex) codimension at least two, and the complement of W in $(J_{k,p})_x$ has codimension at least two. For then we would have

$$O((J_{k,p})_x) = O(W)$$

and

$$O((J_{k,p})_x)^{\mathbb{G}'_{k,p}} = O(W)^{\mathbb{G}'_{k,p}} = O(W/\mathbb{G}'_{k,p}) = O(Z),$$

and it follows that $O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ is finitely generated since Z is affine, and that

$$Z = \text{Spec}(O(Z)) = \text{Spec}(O((J_{k,p})_x)^{\mathbb{G}'_{k,p}}) = ((J_{k,p})_x) // \mathbb{G}'_{k,p}.$$

Similarly if we can find a complex reductive group G containing $\mathbb{G}'_{k,p}$ as a subgroup, and an embedding of $G/\mathbb{G}'_{k,p}$ as an open subset of an affine variety Z with complement of codimension at least two, then $O(G)^{\mathbb{G}'_{k,p}}$ is finitely generated. It follows as above that if Y is any affine variety on which G acts linearly then

$$O(Y)^{\mathbb{G}'_{k,p}} \cong (O(Y) \otimes O(G)^{\mathbb{G}'_{k,p}})^G$$

is finitely generated, and hence so is $O(Y)^{\mathbb{G}_{k,p}} = (O(Y)^{\mathbb{G}'_{k,p}})^{\mathbb{C}^*}$, and similarly $\hat{O}_L(Y)^{\mathbb{G}'_{k,p}}$ and $\hat{O}_L(Y)^{\mathbb{G}_{k,p}}$ are finitely generated if Y is any projective variety with an ample line bundle L on which G acts linearly.

We can use the ideas of [2] to look for suitable affine varieties Z as above, and in particular to prove

Theorem 3.3. *$\mathbb{G}'_{k,p}$ is a subgroup of the special linear group $\text{SL}(\text{sym}^{\leq k} p)$ where*

$$\text{sym}^{\leq k} p = \sum_{i=1}^k \dim \text{Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1}$$

such that the algebra of invariants $O(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}}$ is finitely generated, and every linear action of $\mathbb{G}'_{k,p}$ or $\mathbb{G}_{k,p}$ on an affine or projective variety (with an ample linearisation) which extends to a linear action of $\mathrm{GL}(\mathrm{sym}^{\leq k} p)$ has finitely generated invariants.

Theorem 3.2 is an immediate consequence of this theorem, since the action of $\mathbb{G}_{k,p}$ on $(J_{k,p})_x$ extends to an action of the general linear group $\mathrm{GL}(\mathrm{sym}^{\leq k} p)$. Moreover we will find a geometric description of

$$\mathrm{SL}(\mathrm{sym}^{\leq k} p) // \mathbb{G}'_{k,p} \cong \mathrm{Spec}(O(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}})$$

and thus a geometric description of

$$(J_{k,p})_x // \mathbb{G}'_{k,p} \cong ((J_{k,p})_x \times \mathrm{SL}(\mathrm{sym}^{\leq k} p) // \mathbb{G}'_{k,p}) // \mathrm{SL}(\mathrm{sym}^{\leq k} p).$$

4. A DESCRIPTION VIA TEST CURVES

In [2] the action of \mathbb{G}_k on jet bundles is studied using an idea coming from global singularity theory. The construction goes as follows.

If u, v are positive integers, let $J_k(u, v)$ denote the vector space of k -jets of holomorphic maps $(\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \dots, k$.

With this notation, the fibres of J_k are isomorphic to $J_k(1, n)$, and the group \mathbb{G}_k is simply $J_k(1, 1)$ with the composition action on itself.

If we fix local coordinates z_1, \dots, z_u at $0 \in \mathbb{C}^u$ we can again identify the k -jet of f , using derivatives at the origin, with $(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j \mathbb{C}^u, \mathbb{C}^v)$. This way we get an identification

$$J_k(u, v) = \bigoplus_{j=1}^k \mathrm{Hom}(\mathrm{Sym}^j \mathbb{C}^u, \mathbb{C}^v).$$

We can compose map-jets via substitution and elimination of terms of degree greater than k ; this leads to the composition maps

$$(8) \quad J_k(v, w) \times J_k(u, v) \rightarrow J_k(u, w), \quad (\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1 \text{ modulo terms of degree } > k.$$

When $k = 1$, $J_1(u, v)$ may be identified with u -by- v matrices, and (8) reduces to multiplication of matrices.

The k -jet of a curve $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ is simply an element of $J_k(1, n)$. We call such a curve φ *regular* if $\varphi'(0) \neq 0$. Let us introduce the notation $J_k^{\mathrm{reg}}(1, n)$ for the set of regular curves:

$$J_k^{\mathrm{reg}}(1, n) = \{\gamma \in J_k(1, n); \gamma'(0) \neq 0\}.$$

Note that if $n > 1$ then the complement of $J_k^{\mathrm{reg}}(1, n)$ in $J_k(1, n)$ has codimension at least two. Let $N \geq n$ be any integer and define

$$\Upsilon_k = \{\Psi \in J_k(n, N) : \exists \gamma \in J_k^{\mathrm{reg}}(1, n) : \Psi \circ \gamma = 0\}$$

to be the set of those k -jets which take at least one regular curve to zero. By definition, Υ_k is the image of the closed subvariety of $J_k(n, N) \times J_k^{\mathrm{reg}}(1, n)$ defined by the algebraic

equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Ψ .

This term originally comes from global singularity theory, where this is called the test curve model of A_k -singularities. In global singularity theory singularities of polynomial maps $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^m, 0)$ are classified by their local algebras, and

$$\Sigma_k = \{f \in J_k(n, m) : \mathbb{C}[x_1, \dots, x_n] / \langle f_1, \dots, f_m \rangle \simeq \mathbb{C}[t] / t^{k+1}\}$$

is called a Morin singularity, or A_k -singularity. The test curve model of Gaffney [12] tells us that

$$\overline{\Sigma}_k = \overline{\Upsilon}_k$$

in $J_k(n, m)$.

A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Upsilon_k$, and $\varphi \in J_k^{\text{reg}}(1, 1) = G_k$ is a holomorphic reparametrization of \mathbb{C} , then $\gamma \circ \varphi$ is, again, a test curve of Ψ :

$$(9) \quad \mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N$$

$$\Psi \circ \gamma = 0 \Rightarrow \Psi \circ (\gamma \circ \varphi) = 0.$$

In fact, we get all test curves of Ψ in this way from a single γ if the following open dense property holds: the linear part of Ψ has 1-dimensional kernel. Before stating this more precisely in Proposition 4.3 below, let us write down the equation $\Psi \circ \gamma = 0$ in coordinates in an illustrative case. Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\text{reg}}(1, n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n, N)$ be the k -jets. Using the chain rule, the equation $\Psi \circ \gamma = 0$ reads as follows for $k = 4$:

$$(10) \quad \begin{aligned} \Psi'(\gamma') &= 0, \\ \frac{1}{2!} \Psi''(\gamma'') + \Psi'''(\gamma', \gamma') &= 0, \\ \frac{1}{3!} \Psi''(\gamma''') + \frac{2}{2!} \Psi''(\gamma', \gamma'') + \Psi'''(\gamma', \gamma', \gamma') &= 0, \\ \frac{1}{4!} \Psi'(\gamma''''') + \frac{2}{3!} \Psi''(\gamma', \gamma''') + \frac{1}{2!2!} \Psi''(\gamma'', \gamma'') + \frac{3}{2!} \Psi'''(\gamma', \gamma', \gamma'') + \Psi''''(\gamma', \gamma', \gamma', \gamma') &= 0. \end{aligned}$$

Definition 4.1. To simplify our formulas we introduce the following notation for a partition $\tau = [i_1 \dots i_l]$ of the integer $i_1 + \dots + i_l$:

- the *length*: $|\tau| = l$,
- the *sum*: $\sum \tau = i_1 + \dots + i_l$,
- the *number of permutations*: $\text{perm}(\tau)$ is the number of different sequences consisting of the numbers i_1, \dots, i_l (e.g. $\text{perm}([1, 1, 1, 3]) = 4$),
- $\gamma_\tau = \prod_{j=1}^l \gamma^{(i_j)} \in \text{Sym}^l \mathbb{C}^n$ and $\Psi(\gamma_\tau) = \Psi^l(\gamma^{(i_1)}, \dots, \gamma^{(i_l)}) \in \mathbb{C}^N$.

Lemma 4.2. Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\text{reg}}(1, n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n, N)$ be k -jets. Then the equation $\Psi \circ \gamma = 0$ is equivalent to the following system of k linear

equations with values in \mathbb{C}^N :

$$(11) \quad \sum_{\tau \in \Pi[m]} \frac{\text{perm}(\tau)}{\prod_{i \in \tau} i!} \Psi(\gamma_\tau) = 0, \quad m = 1, 2, \dots, k,$$

where $\Pi[m]$ denotes the set of all partitions of m .

For a given $\gamma \in J_k^{\text{reg}}(1, n)$ let \mathcal{S}_γ denote the set of solutions of (11); that is,

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n, N); \Psi \circ \gamma = 0\}.$$

The equations (11) are linear in Ψ , hence

$$\mathcal{S}_\gamma \subset J_k(n, N)$$

is a linear subspace of codimension kN . Moreover, the following holds:

Proposition 4.3. ([2], Proposition 4.4)

- (i) For $\gamma \in J_k^{\text{reg}}(1, n)$, the set of solutions $\mathcal{S}_\gamma \subset J_k(n, N)$ is a linear subspace of codimension kN .
- (ii) Set

$$J_k^o(n, N) = \{\Psi \in J_k(n, N) \mid \dim \ker(\Psi') = 1\}.$$

For any $\gamma \in J_k^{\text{reg}}(1, n)$, the subset $\mathcal{S}_\gamma \cap J_k^o(n, N)$ of \mathcal{S}_γ is dense.

- (iii) If $\Psi \in J_k^o(n, N)$, then Ψ belongs to at most one of the spaces \mathcal{S}_γ . More precisely,

$$\text{if } \gamma_1, \gamma_2 \in J_k^{\text{reg}}(1, n), \quad \Psi \in J_k^o(n, N) \text{ and } \Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0,$$

then there exists $\varphi \in J_k^{\text{reg}}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

- (iv) Given $\gamma_1, \gamma_2 \in J_k^{\text{reg}}(1, n)$, we have $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ if and only if there is some $\varphi \in J_k^{\text{reg}}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

By the second part of Proposition 4.3 we have a well-defined map

$$\nu : J_k^{\text{reg}}(1, n) \rightarrow \text{Grass}(\text{codim} = kN, J_k(n, N)), \quad \gamma \mapsto \mathcal{S}_\gamma$$

to the Grassmannian of codimension- kN subspaces in $J_k(n, N)$. From the last part of Proposition 4.3 it follows that:

Proposition 4.4. ([2]) ν is \mathbb{G}_k -invariant on the $J_k^{\text{reg}}(1, 1)$ -orbits, and the induced map on the orbits

$$(12) \quad \bar{\nu} : J_k^{\text{reg}}(1, n)/\mathbb{G}_k \hookrightarrow \text{Grass}(\text{codim} = kN, J_k(n, N))$$

is injective.

5. EMBEDDING INTO THE FLAG OF EQUATIONS

In this section we will recast the embedding (12) of $J_k^{\text{reg}}(1, n)/\mathbb{G}_k$ given by Proposition 4.4 into a more useful form, still following [2]. Let us rewrite the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in J_k^{\text{reg}}(1, n)$ in a dual form. The system is based on the standard composition map (8):

$$J_k(n, N) \times J_k(1, n) \longrightarrow J_k(1, N),$$

which, via the identification $J_k(n, N) = J_k(n, 1) \otimes \mathbb{C}^N$, is derived from the map

$$J_k(n, 1) \times J_k(1, n) \longrightarrow J_k(1, 1)$$

via tensoring with \mathbb{C}^N . Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$(13) \quad \phi : J_k(1, n) \longrightarrow \text{Hom}(J_k(1, 1)^*, J_k(n, 1)^*).$$

If $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k(1, n) = (\mathbb{C}^n)^k$ is the k -jet of a curve, we can put $\gamma^{(j)} \in \mathbb{C}^n$ into the j th column of an $n \times k$ matrix, and

- identify $J_k(1, n)$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$;
- identify $J_k(n, 1)^*$ with $\text{Sym}^{\leq k} \mathbb{C}^n = \bigoplus_{l=1}^k \text{Sym}^l \mathbb{C}^n$;
- identify $J_k(1, 1)^*$ with \mathbb{C}^k .

Using these identifications, we can recast the map ϕ in (13) as

$$(14) \quad \phi_k : \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \longrightarrow \text{Hom}(\mathbb{C}^k, \text{Sym}^{\leq k} \mathbb{C}^n),$$

which may be written out explicitly as follows

$$(\gamma', \gamma'', \dots, \gamma^{(k)}) \mapsto \left(\gamma', \gamma'' + (\gamma')^2, \dots, \sum_{i_1+i_2+\dots+i_s=d} \frac{1}{i_1! \dots i_s!} \gamma^{(i_1)} \gamma^{(i_2)} \dots \gamma^{(i_s)} \right).$$

The set of solutions \mathcal{S}_γ is the linear subspace orthogonal to the image of $\phi_k(\gamma', \dots, \gamma^{(k)})$ tensored by \mathbb{C}^N ; that is,

$$\mathcal{S}_\gamma = \text{im}(\phi_k(\gamma))^\perp \otimes \mathbb{C}^N \subset J_k(n, N).$$

Consequently, it is straightforward to take $N = 1$ and define

$$(15) \quad \mathcal{S}_\gamma = \text{im}(\phi_k(\gamma)) \in \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n).$$

Moreover, let $B_k \subset GL(k)$ denote the Borel subgroup consisting of upper triangular matrices and let

$$\text{Flag}_k(\mathbb{C}^n) = \text{Hom}(\mathbb{C}^k, \text{Sym}^{\leq k} \mathbb{C}^n)/B_k = \{0 = F_0 \subset F_1 \subset \dots \subset F_k \subset \mathbb{C}^n, \dim F_l = l\}$$

denote the full flag of k -dimensional subspaces of $\text{Sym}^{\leq k} \mathbb{C}^n$. In addition to (15) we can analogously define

$$(16) \quad \mathcal{F}_\gamma = (\text{im}(\phi(\gamma^1)) \subset \text{im}(\phi(\gamma^2)) \subset \dots \subset \text{im}(\phi(\gamma^k))) \in \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n).$$

Using these definitions Proposition 4.3 implies the the following version of Proposition 4.4, which does not contain the parameter N .

Proposition 5.1. *The map ϕ in (14) is a \mathbb{G}_k -invariant algebraic morphism*

$$\phi : J_k^{\text{reg}}(1, n) \rightarrow \text{Hom}(\mathbb{C}^k, \text{Sym}^{\leq k} \mathbb{C}^n),$$

which induces

- an injective map on the \mathbb{G}_k -orbits to the Grassmannian:

$$\phi^{Gr} : J_k^{\text{reg}}(1, n)/\mathbb{G}_k \hookrightarrow \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$$

defined by $\phi^{Gr}(\gamma) = S_\gamma$;

- an injective map on the \mathbb{G}_k -orbits to the flag manifold:

$$\phi^{Flag} : J_k^{\text{reg}}(1, n)/G_k \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

defined by $\phi^{Flag}(\gamma) = \mathcal{F}_\gamma$.

In addition,

$$\phi^{Gr} = \phi^{Flag} \circ \pi_k$$

where $\pi_k : \text{Flag}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \rightarrow \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ is the projection to the k -dimensional subspace.

Composing ϕ^{Gr} with the Plücker embedding

$$\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \text{Sym}^{\leq k} \mathbb{C}^n)$$

we get an embedding

$$(17) \quad \phi^{\text{Proj}} : J_k^{\text{reg}}(1, n)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^n)).$$

The image

$$\phi^{Gr}(J_k^{\text{reg}}(1, n))/\mathbb{G}_k \subset \text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$$

is a $GL(n)$ -orbit in $\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$, and therefore a nonsingular quasi-projective variety. Its closure is, however, a highly singular subvariety of $\text{Grass}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$, which when $k \leq n$ is a finite union of $GL(n)$ orbits.

Definition 5.2. Recall that we can identify $J_k(1, n)$ with $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and then

$$J_k^{\text{reg}}(1, n) = \{\rho \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \rho(e_1) \neq 0\}.$$

Let

$$J_k^{\text{nondeg}}(1, n) = \{\rho \in \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) : \text{rank} \rho = \max\{k, n\}\}$$

and let

$$X_{n,k} = \phi^{\text{Proj}}(J_k^{\text{nondeg}}(1, n)), \quad Y_{n,k} = \phi^{\text{Proj}}(J_k^{\text{reg}}(1, n)),$$

so that if $n \leq k$ then

$$X_{n,k} \subset Y_{n,k} \subset \text{Grass}(n, \text{Sym}^{\leq k} \mathbb{C}^n) \subset \mathbb{P}(\wedge^k (\text{Sym}^{\leq k} \mathbb{C}^n)).$$

It is clear that $J_k^{\text{nondeg}}(1, n)$ is an open subset of $J_k^{\text{reg}}(1, n)$. If we identify the elements of $J_k(1, n)$ with $n \times k$ matrices whose columns are the derivatives of the map germs $f = (f', \dots, f^{(n)}) : \mathbb{C} \rightarrow \mathbb{C}^n$, then $J_k^{\text{nondeg}}(1, n)$ is the set of such matrices of maximal rank and $J_k^{\text{reg}}(1, n)$ consists of the matrices with nonzero first column.

Definition 5.3. Let e_1, \dots, e_n be the standard basis of \mathbb{C}^n ; then

$$\{e_{i_1, i_2, \dots, i_s} = e_{i_1} \dots e_{i_s} : 1 \leq i_1 \leq \dots \leq i_s \leq n, 1 \leq s \leq k\}$$

is a basis of $\text{Sym}^{\leq k} \mathbb{C}^n$, and

$$\{e_{\varepsilon_1} \wedge \dots \wedge e_{\varepsilon_n} : \varepsilon_l \in \Pi_{\leq n}\}$$

is a basis of $\mathbb{P}(\wedge^n(\text{Sym}^{\leq k} \mathbb{C}^n))$, where

$$\Pi_{\leq n} = \{(i_1, i_2, \dots, i_s) : 1 \leq i_1 \leq \dots \leq i_s \leq n, 1 \leq s \leq k\}.$$

The corresponding coordinates of $x \in \text{Sym}^{\leq k} \mathbb{C}^n$ will be denoted by $x_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d}$. Let $A_{n,k} \subset \mathbb{P}(\wedge^k(\text{Sym}^{\leq k} \mathbb{C}^n))$ consist of the points whose projection to $\wedge^k(\mathbb{C}^n)$ is nonzero. This is the subset where $x_{i_1, i_2, \dots, i_k} \neq 0$ for some $1 \leq i_1 \leq \dots \leq i_k \leq n$.

Remark 5.4. If $n = k$ then $A_{n,n} \subset \mathbb{P}(\wedge^k(\text{Sym}^{\leq k} \mathbb{C}^n))$ is the affine chart where $x_{1,2,\dots,n} \neq 0$.

Let us take a closer look at the space $\text{Grass}(n, \text{Sym}^{\leq k} \mathbb{C}^n)$, which has an induced $\text{GL}(n)$ action coming from the $\text{GL}(n)$ action on $\text{Sym}^{\leq k} \mathbb{C}^n$. Since ϕ^{Proj} is a $\text{GL}(n)$ -equivariant embedding, we conclude that

Lemma 5.5. (i) For $k \leq n$ $X_{n,k}$ is the $\text{GL}(n)$ orbit of

$$(18) \quad \mathbf{z} = \phi^{\text{Proj}}(e_1, \dots, e_k) = [e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s})]$$

in $\mathbb{P}(\wedge^k(\text{Sym}^{\leq k} \mathbb{C}^n))$. For arbitrary $g \in \text{GL}(n)$ with column vectors v_1, \dots, v_n the action is given by

$$g \cdot \mathbf{z} = \phi^{\text{Proj}}(g) = \phi^{\text{Proj}}(v_1, \dots, v_n) = [v_1 \wedge (v_2 + v_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=n} v_{i_1} \dots v_{i_s})].$$

(ii) For $k \leq n$ $Y_{n,k}$ is a finite union of $\text{GL}(n)$ orbits.

(iii) For $k > n$ the images $X_{n,k}$ and $Y_{n,k}$ are $\text{GL}(n)$ -invariant quasi-projective varieties with no dense $\text{GL}(n)$ orbit.

Lemma 5.6. If $k \leq n$ then

(i) $A_{n,k}$ is invariant under the $\text{GL}(n)$ action on $\mathbb{P}(\wedge^k(\text{Sym}^{\leq k} \mathbb{C}^n))$.

(ii) $X_{n,k} \subset A_{n,k}$; however, $Y_{n,k} \not\subset A_{n,k}$.

Proof. To prove the first part take a lift

$$\tilde{z} = \tilde{z}^1 \oplus \tilde{z}^2 \in \text{Hom}(\mathbb{C}^n, \text{Sym}^{\leq k} \mathbb{C}^n)$$

of $z \in \text{Grass}(n, \text{Sym}^{\leq k} \mathbb{C}^n)$, where

$$z^1 \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \text{ and } z^2 \in \text{Hom}(\mathbb{C}^n, \oplus_{i=2}^n \text{Sym}^i(\mathbb{C}^n))$$

Then $z \in A_{n,k}$ if and only if $x_{1,2,\dots,n}(z) = \det(\tilde{z}^1) \neq 0$, which is preserved by the $\text{GL}(n)$ action. For the second part note that for $(v_1, \dots, v_k) \in J_k^{\text{nondeg}}(1, n)$ we have $v_1 \wedge \dots \wedge v_k \neq 0$ so by definition $\phi^{\text{Proj}}(v_1, \dots, v_k) \in A_{n,k}$. On the other hand

$$\phi^{\text{Proj}}(e_1, 0, \dots, 0) = e_1 \wedge e_1^2 \wedge \dots \wedge e_1^k \in Y_{n,k} \setminus A_{n,k}.$$

□

When $k = n$ we have

Lemma 5.7. $X_{k,k} \cong \mathrm{GL}(k)/\mathbb{G}_k$ is embedded in the affine space $A_{k,k} \subset \mathbb{P}(\wedge^k \mathrm{Sym}^{\leq k} \mathbb{C}^k)$ as the $\mathrm{GL}(k)$ orbit of $[e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s})]$.

6. AFFINE EMBEDDINGS OF $\mathrm{SL}(k)/\mathbb{U}_k$

In the last section we embedded $\mathrm{GL}(k)/\mathbb{G}_k$ in the affine space $A_{k,k} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k))$ as the $\mathrm{GL}(k)$ orbit of

$$[e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s})] \in \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)).$$

Equivalently we have

$$\mathrm{SL}(k)/\mathrm{SL}(k) \cap \mathbb{G}_k = \mathrm{SL}(k)/\mathbb{U}_k \rtimes F_k$$

embedded in $\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)$ as the $\mathrm{SL}(k)$ orbit of

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s}),$$

where $\mathrm{SL}(k) \cap \mathbb{G}_k$ is the semi-direct product $\mathbb{U}_k \rtimes F_k$ of \mathbb{U}_k by the finite group F_k of ℓ_k th roots of unity in \mathbb{C} for $\ell_k = 1 + \dots + k = \binom{k+1}{2}$, embedded in $\mathrm{SL}(k)$ as

$$\epsilon \mapsto \begin{pmatrix} \epsilon & 0 & \dots & 0 \\ 0 & \epsilon^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \epsilon^k \end{pmatrix} \in \mathrm{SL}(k).$$

In this section we will look for affine embeddings of $\mathrm{SL}(k)/\mathbb{U}_k$ in spaces of the form

$$W_{k,K} = \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

for suitable K and study their closures.

Lemma 6.1. Let $K = M(1 + 2 + \dots + k) + 1 = \binom{k+1}{2}M + 1$ where $M \in \mathbb{N}$. Then the point

$$p_k \otimes e_1^{\otimes K} \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

where

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s}) \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)$$

has stabiliser \mathbb{U}_k in $\mathrm{SL}(k)$.

Proof. By Proposition 5.1 the stabiliser of

$$[p_k] \in \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)) \cong \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}e_1)^{\otimes K}) \subseteq \mathbb{P}(W_{k,K})$$

in $GL(k)$ is $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$, so the stabiliser of

$$p_k \otimes e_1^{\otimes K} \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

is contained in \mathbb{G}_k . Moreover by the proof of Proposition 5.1 the stabiliser of $p_k \otimes e_1^{\otimes K}$ contains \mathbb{U}_k . Finally

$$\begin{pmatrix} z & 0 & \dots & 0 \\ 0 & z^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & z^k \end{pmatrix} \in \mathbb{C}^* \subseteq \mathbb{G}_k$$

acts on $p_k \otimes e_1^{\otimes K}$ as multiplication by

$$z^{1+2+\dots+k+K} = z^{(M+1)(1+2+\dots+k)+1}$$

and has determinant 1 if and only if $z^{1+2+\dots+k} = 1$, so it lies in $SL(k)$ and fixes $p_k \otimes e_1^{\otimes K}$ if and only if $z = 1$. \square

We will prove

Theorem 6.2. *If $k \geq 4$ and $K = M(1 + 2 + \dots + k) + 1$ where $M \in \mathbb{N}$ is sufficiently large, then the orbit of $p_k \otimes e_1^{\otimes K}$ where*

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge \left(\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s} \right) \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)$$

under the natural action of $SL(k)$ on

$$W_{k,K} = \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

is isomorphic to $SL(k)/\mathbb{U}_k$, and its complement in its closure $\overline{SL(k)(p_k \otimes e_1^{\otimes K})}$ in $W_{k,K}$ has codimension at least two.

This theorem has an immediate corollary.

Corollary 6.3. *If $k \geq 2$ then \mathbb{U}_k is a Grosshans subgroup of $SL(k)$, so that every linear action of \mathbb{U}_k which extends to a linear action of $SL(k)$ has finitely generated invariants.*

Proof. This follows directly from Theorem 6.2 when $k \geq 4$. When $k = 2$ and $k = 3$ it is already known (cf. [27]). \square

The remainder of this section will be devoted to proving Theorem 6.2.

It follows directly from Lemma 6.1 that the $SL(k)$ -orbit of $p_k \otimes e_1^{\otimes K}$ in $W_{k,K} = \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$ is isomorphic to $SL(k)/\mathbb{U}_k$.

Recall that

$$\mathbb{U}_k = \left\{ \begin{pmatrix} 1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & 1 & 2\alpha_2 & \cdots & 2\alpha_{k-1} + \dots \\ 0 & 0 & 1 & \cdots & 3\alpha_{k-2} + \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & (k-1)\alpha_2 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \alpha_2, \dots, \alpha_k \in \mathbb{C} \right\}$$

so that \mathbb{U}_k is generated along its last column as well as along its first row.

Let $B_k \subset \mathrm{SL}(k)$ denote the standard Borel subgroup of $\mathrm{SL}(k)$ which stabilises the filtration $\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \cdots \subset \mathbb{C}^k$. Then $B_k = B_{k-1} \cdot \mathbb{U}_k$ where the Borel subgroup B_{k-1} of $\mathrm{GL}(k-1) = \mathrm{GL}(\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{k-1})$ is embedded diagonally in $\mathrm{SL}(k)$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}.$$

Since \mathbb{U}_k stabilises p_k and e_1 we have

$$\overline{B_k(p_k \otimes e_1^{\otimes K})} = \overline{B_{k-1}(p_k \otimes e_1^{\otimes K})},$$

and since $\mathrm{SL}(k)/B_k$ is projective we have

$$\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})} = \overline{\mathrm{SL}(k)B_k(p_k \otimes e_1^{\otimes K})} = \overline{\mathrm{SL}(k)B_{k-1}(p_k \otimes e_1^{\otimes K})}.$$

Since the closure $\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ of the $\mathrm{SL}(k)$ -orbit of $p_k \otimes e_1^{\otimes K}$ in $W_{k,K}$ is the union of finitely many $\mathrm{SL}(k)$ -orbits, to prove Theorem 6.2 it suffices to prove

Lemma 6.4. *Suppose that $k \geq 4$ and a and b are strictly positive integers with b/a large enough and that x lies in the closure in*

$$(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the Borel subgroup B_k of $\mathrm{SL}(k)$. Then either $x \in B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k+1$.

We will split the proof of this lemma into two parts. Let T_k denote the standard maximal torus of $\mathrm{SL}(k)$ consisting of the diagonal matrices in $\mathrm{SL}(k)$. Lemma 6.4 follows immediately from Lemmas 6.5 and 6.6 below.

Lemma 6.5. *Suppose that $k \geq 4$ and a and b are strictly positive integers with b/a large enough and that x lies in the closure $\overline{T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})}$ in*

$$(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the maximal torus T_k of $\mathrm{SL}(k)$. Then either $x \in T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k+1$.

Lemma 6.6. *Suppose that $k \geq 2$ and a and b are strictly positive integers and that x lies in the closure in*

$$(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $\overline{p_k^{\otimes a} \otimes e_1^{\otimes b}}$ under the natural action of the Borel subgroup B_k of $\mathrm{SL}(k)$. Then either $x \in B_k T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k + 1$.

We will start with the proof of Lemma 6.6.

Proof. We have

$$x \in \overline{B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})} = \overline{B_{k-1}(p_k^{\otimes a} \otimes e_1^{\otimes b})}$$

as above, so there is a sequence of matrices

$$b^{(m)} = \begin{pmatrix} b_{11}^{(m)} & b_{12}^{(m)} & \cdots & b_{1k-1}^{(m)} & 0 \\ 0 & b_{22}^{(m)} & \cdots & b_{2k-1}^{(m)} & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 0 & b_{kk}^{(m)} \end{pmatrix} \in B_{k-1} \subset \mathrm{SL}(k)$$

such that $b^{(m)}(p_k^{\otimes a} \otimes e_1^{\otimes b}) \rightarrow x$ as $m \rightarrow \infty$. Now expanding the wedge product in the definition of p_k we get

$$b^{(m)}(p_k^{\otimes a}) = (e_1 \wedge \cdots \wedge e_n + \cdots + (b_{11}^{(m)})^{1+2+\cdots+k} e_1 \otimes e_1^2 \otimes \cdots \otimes e_1^k)^{\otimes a}$$

while

$$b^{(m)}(e_1^{\otimes b}) = (b_{11}^{(m)})^b e_1^{\otimes b},$$

so by considering the coefficient of $(e_1 \wedge \cdots \wedge e_n)^{\otimes a} \otimes e_1^{\otimes b}$ we see that $(b_{11}^{(m)})^b$ tends to a limit in \mathbb{C} as $m \rightarrow \infty$. Thus, by replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$b_{11}^{(m)} \rightarrow b_{11}^{(\infty)} \in \mathbb{C}$$

as $m \rightarrow \infty$.

First suppose that $k = 2$. Then $\mathrm{Sym}^{\leq k}\mathbb{C}^k = \mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2$ and

$$(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b} = (\wedge^2(\mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2))^{\otimes a} \otimes (\mathbb{C}^2)^{\otimes b}$$

and

$$p_k = e_1 \wedge (e_2 + e_1^2),$$

so if

$$b^{(m)} = \begin{pmatrix} b_{11}^{(m)} & b_{12}^{(m)} \\ 0 & b_{22}^{(m)} \end{pmatrix} \in \mathrm{SL}(2)$$

then $b_{11}^{(m)} b_{22}^{(m)} = 1$ and

$$\begin{aligned} b^{(m)}(p_2^{\otimes a} \otimes e_1^{\otimes b}) &= (b_{11}^{(m)})^b (e_1 \wedge (e_2 + (b_{11}^{(m)})^3 e_1^2))^{\otimes a} \otimes e_1^{\otimes b} \\ &\rightarrow x = (b_{11}^{(\infty)})^b (e_1 \wedge (e_2 + (b_{11}^{(\infty)})^3 e_1^2))^{\otimes a} \otimes e_1^{\otimes b} \end{aligned}$$

as $m \rightarrow \infty$. If $b_{11}^{(\infty)} \neq 0$ then $x \in \mathrm{SL}(2)((p_2^{\otimes a} \otimes e_1^{\otimes b}))$, while if $b_{11}^{(\infty)} = 0$ then $x = 0$ is fixed by $\mathrm{SL}(2)$ which has dimension $3 = k + 1$.

Now suppose that $k > 2$, and assume first that $b_{11}^{(\infty)} \neq 0$. We have that

$$b^{(m)}(p_k^{\otimes a} \otimes e_1^{\otimes b}) = (b_{11}^{(m)})^b (b^{(m)} p_k)^{\otimes a} \otimes e_1^{\otimes b} \rightarrow x$$

and $b_{11}^{(m)} \rightarrow b_{11}^{(\infty)} \in \mathbb{C} \setminus \{0\}$ as $m \rightarrow \infty$, so by replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$(b_{11}^{(m)})^{b/a} b^{(m)} p_k \rightarrow p_k^\infty \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)$$

as $m \rightarrow \infty$, where

$$(19) \quad b^{(m)} p_k = b_{11}^{(m)} e_1 \wedge (b_{22}^{(m)} e_2 + (b_{11}^{(m)})^2 e_1^2) \wedge \dots \wedge (b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \dots \\ \dots + b_{1i}^{(m)} e_1 + \sum_{s=2}^{i-1} \sum_{i_1 + \dots + i_s = i} (b_{i_1 i_1}^{(m)} e_{i_1} + \dots + b_{1 i_1}^{(m)} e_1) \dots (b_{i_s i_s}^{(m)} e_{i_s} + \dots + b_{1 i_s}^{(m)} e_1) + (b_{11}^{(m)})^i e_1^i) \wedge \dots$$

Looking at the coefficient of

$$e_1 \wedge e_1^2 \wedge \dots \wedge e_1^{i-1} \wedge e_j \wedge e_1^{i+1} \wedge \dots \wedge e_1^k$$

when $1 \leq j \leq i \leq k$, we see that

$$(b_{11}^{(m)})^{1+2+\dots+(i-1)+(i+1)+\dots+k} b_{ji}^{(m)}$$

tends to a limit in \mathbb{C} as $m \rightarrow \infty$, and so since $b_{11}^{(\infty)} \neq 0$

$$b_{ji}^{(m)} \rightarrow b_{ji}^{(\infty)} \in \mathbb{C}.$$

Also $b_{11}^{(m)} b_{22}^{(m)} \dots b_{kk}^{(m)} = 1$ for all m , so $b_{11}^{(\infty)} b_{22}^{(\infty)} \dots b_{kk}^{(\infty)} = 1$, so $b^{(m)} \rightarrow b^{(\infty)} \in \mathrm{SL}(k)$. Therefore

$$x = b^{(\infty)}(p_k^{\otimes a} \otimes e_1^{\otimes b})$$

lies in the orbit of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ as required.

So it remains to consider the case when $b_{11}^{(\infty)} = 0$. If $p_k^\infty = 0$ then its stabiliser is $\mathrm{SL}(k)$ which has dimension $k^2 - 1 \geq k + 1$, so we can assume that $p_k^\infty \neq 0$. Recall that then

$$(b_{11}^{(m)})^{b/a} b^{(m)} p_k \rightarrow p_k^\infty \in \wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k)$$

and

$$[b^{(m)} p_k] \rightarrow [p_k^\infty] \in \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k} \mathbb{C}^k))$$

as $m \rightarrow \infty$, where

$$b^{(m)} p_k = b_{11}^{(m)} e_1 \wedge (b_{22}^{(m)} e_2 + (b_{11}^{(m)})^2 e_1^2) \wedge \dots \wedge (b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \dots \\ \dots + b_{1i}^{(m)} e_1 + \sum_{s=2}^{i-1} \sum_{i_1 + \dots + i_s = i} (b_{i_1 i_1}^{(m)} e_{i_1} + \dots + b_{1 i_1}^{(m)} e_1) \dots (b_{i_s i_s}^{(m)} e_{i_s} + \dots + b_{1 i_s}^{(m)} e_1) + (b_{11}^{(m)})^i e_1^i) \wedge \dots$$

By replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$[b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \dots + b_{1i}^{(m)} e_1] \rightarrow [c_{ii}^{(\infty)} e_i + c_{i-1i}^{(\infty)} e_{i-1} + \dots + c_{1i}^{(\infty)} e_1] \in \mathbb{P}(\mathbb{C}^k)$$

as $m \rightarrow \infty$ for $2 \leq i \leq k$, which implies that

$$\begin{aligned} & [(b_{i_1 i_1}^{(m)} e_{i_1} + \dots + b_{1 i_1}^{(m)} e_1) \dots (b_{i_s i_s}^{(m)} e_{i_s} + \dots + b_{1 i_s}^{(m)} e_1)] \rightarrow \\ & [(c_{i_1 i_1}^{(\infty)} e_{i_1} + \dots + c_{1 i_1}^{(\infty)} e_1) \dots (c_{i_s i_s}^{(\infty)} e_{i_s} + \dots + c_{1 i_s}^{(\infty)} e_1)] \in \mathbb{P}(\text{Sym}^i \mathbb{C}^k) \end{aligned}$$

whenever $i_1 + \dots + i_s = i \in \{2, \dots, k\}$, and hence that

$$p_k^\infty \in \wedge^k(\text{Sym}^{\leq k} D)$$

where D is the span in \mathbb{C}^k of

$$\{e_1\} \cup \{c_{ii}^{(\infty)} e_i + c_{i-1i}^{(\infty)} e_{i-1} + \dots + c_{1i}^{(\infty)} e_1 : 2 \leq i \leq k\}.$$

Moreover since $b^{(m)} \in B_{k-1}$ we have $b_{jk}^{(m)} = 0$ if $j < k$ so

$$[c_{kk}^{(\infty)} e_k + c_{k-1k}^{(\infty)} e_{k-1} + \dots + c_{1k}^{(\infty)} e_1] = [e_k]$$

so $e_k \in D$.

Note that $b^{(m)} \in B_{k-1}$ and B_{k-1} normalises the maximal unipotent subgroup U_k of B_k which contains the stabiliser \mathbb{U}_k of p_k . Therefore for each m there is a $(k-1)$ -dimensional subgroup of U_k which stabilises $b^{(m)} p_k$, and it follows that there is a $(k-1)$ -dimensional subgroup \mathbb{U}_k^∞ of U_k which stabilises p_k^∞ . In addition by [3] Theorem 6.4 if p_k^∞ does not lie in $\text{SL}(k)p_k$ then it is stabilised by a nontrivial one-parameter subgroup $\lambda^\infty : \mathbb{C}^* \rightarrow \text{SL}(k)$ of $\text{SL}(k)$. Moreover if $D \neq \mathbb{C}^k$ then there is some $j \in \{2, \dots, k-1\}$ such that e_j is not in D , and then there is an automorphism of \mathbb{C}^k which fixes every element of D and sends e_j to $e_j + e_k$. This automorphism is independent of \mathbb{U}_k^∞ (since $\mathbb{U}_k^\infty \subseteq U_k$) and the one-parameter subgroup λ^∞ of $\text{SL}(k)$ fixing p_k^∞ , so the stabiliser of p_k^∞ in $\text{SL}(k)$ has dimension at least

$$\dim \mathbb{U}_k^\infty + 2 = k + 1.$$

Thus we can assume that $D = \mathbb{C}^k$, and hence $c_{ii}^{(\infty)} \neq 0$ for $2 \leq i \leq k$, so that

$$\frac{b_{ji}^{(m)}}{b_{ii}^{(m)}} \rightarrow \frac{c_{ji}^{(\infty)}}{c_{ii}^{(\infty)}} \in \mathbb{C}$$

as $m \rightarrow \infty$. Then by applying an element of B_{k-1} to p_k^∞ we can assume that

$$[c_{ii}^{(\infty)} e_i + c_{i-1i}^{(\infty)} e_{i-1} + \dots + c_{1i}^{(\infty)} e_1] = [e_i]$$

or equivalently that

$$[b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \dots + b_{1i}^{(m)} e_1] \rightarrow [e_i]$$

as $m \rightarrow \infty$ for $2 \leq i \leq k$, and hence that

$$[(b_{i_1 i_1}^{(m)} e_{i_1} + \dots + b_{1 i_1}^{(m)} e_1) \dots (b_{i_s i_s}^{(m)} e_{i_s} + \dots + b_{1 i_s}^{(m)} e_1)] \rightarrow [e_{i_1} \dots e_{i_s}] \in \mathbb{P}(\text{Sym}^i \mathbb{C}^k)$$

whenever $i_1 + \dots + i_s = i \in \{2, \dots, k\}$. Now by again replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$[b_{ii}^{(m)} e_i + b_{i-1i}^{(m)} e_{i-1} + \dots + b_{1i}^{(m)} e_1 + \sum_{s=2}^{i-1} \sum_{i_1+\dots+i_s=i} (b_{i_1 i_1}^{(m)} e_{i_1} + \dots + b_{1 i_1}^{(m)} e_1)] \rightarrow [d_i^\infty] \in \mathbb{P}(\text{Sym}^{\leq k} \mathbb{C}^k)$$

where

$$d_i^\infty = \gamma_i^{(\infty)} e_i + \sum_{s=2}^i \sum_{i_1+\dots+i_s=i} \gamma_{i_1 \dots i_s}^{(\infty)} e_{i_1} \dots e_{i_s} \in \text{Sym}^{\leq k} \mathbb{C}^k \setminus \{0\}$$

for some $\gamma_{i_1 \dots i_s}^{(\infty)} \in \mathbb{C}$. In addition $\{d_i^\infty : 1 \leq i \leq k\}$ is linearly independent so that

$$[p_k^\infty] = [d_1^\infty \wedge \dots \wedge d_k^\infty] \in \mathbb{P}(\wedge^k(\text{Sym}^{\leq k} \mathbb{C}^k))$$

and $p_k^\infty = \lim_{m \rightarrow \infty} t^{(m)} p_k$ where $t^{(m)}$ is the diagonal matrix with entries $b_{11}^{(m)}, \dots, b_{kk}^{(m)}$.

Thus we can assume that $p_k^\infty \in \overline{T_k p_k}$ where T_k is the standard maximal torus in $\text{SL}(k)$, which completes the proof of Lemma 6.6. \square

It therefore remains to prove Lemma 6.5. We can continue with the notation above and use the following standard result:

Lemma 6.7. *Let T be an algebraic torus acting on the projective variety Z , and $z \in Z$. Then $y \in \overline{Tz}$ if and only if there is $\tau \in T$, and a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow T$ such that $\tau y \in \overline{\lambda(\mathbb{C}^*)z}$.*

Hence we may assume without loss of generality that there is a one-parameter subgroup

$$t \mapsto \lambda(t) = \begin{pmatrix} t^{\lambda_1} & 0 & \dots & 0 \\ 0 & t^{\lambda_2} & 0 & \dots \\ & & \dots & \\ 0 & \dots & & 0 & t^{\lambda_k} \end{pmatrix}$$

of $\text{SL}(k)$ such that $\lambda_1 > 0$ and $t^{\lambda_1 b/a} \lambda(t) p_k \rightarrow p_k^\infty$ as $t \rightarrow 0$. Therefore

$$p_k^\infty = \lim_{t \rightarrow 0} t^{\lambda_1 b/a} e_1 \wedge (e_2 + t^{2\lambda_1 - \lambda_2} e_1^2) \wedge \dots \wedge (e_k + \sum_{s=2}^k \sum_{i_1+\dots+i_s=s} i_1 + \dots + i_s = k t^{\lambda_{i_1} + \dots + \lambda_{i_s} - \lambda_k} e_{i_1} \dots e_{i_s})$$

where $\lambda_1 + \dots + \lambda_k = 0$. We are assuming that $p_k^\infty \neq 0$ so

$$[p_k^\infty] = \lim_{t \rightarrow 0} [e_1 \wedge (e_2 + t^{2\lambda_1 - \lambda_2} e_1^2) \wedge \dots \wedge (e_k + \sum_{s=2}^k \sum_{i_1+\dots+i_s=s} i_1 + \dots + i_s = k t^{\lambda_{i_1} + \dots + \lambda_{i_s} - \lambda_k} e_{i_1} \dots e_{i_s})].$$

If $\lambda_{i_1} + \dots + \lambda_{i_s} < \lambda_j$ for some $j \in \{2, \dots, k-1\}$ and $s \geq 2$ and $i_1, \dots, i_s \geq 1$ such that $i_1 + \dots + i_s = j$, then $[p_k^\infty]$ is independent of e_j and so as above the stabiliser of p_k^∞ in $\text{SL}(k)$ has dimension at least $k+1$. So we can assume that

$$(20) \quad \lambda_{i_1} + \dots + \lambda_{i_s} \geq \lambda_j$$

for any $j \in \{2, \dots, k-1\}$ and $s \geq 2$ and $i_1, \dots, i_s \geq 1$ such that $i_1 + \dots + i_s = j$, and in particular that $\lambda_j \leq j\lambda_1$ for each $j \in \{2, \dots, k-1\}$. Let

$$(21) \quad \rho_j = j\lambda_1 - \lambda_j$$

for $j \in \{1, \dots, k-1\}$; then $\rho_1 = 0$ and $\rho_j \geq 0$ and

$$\rho_{i_1} + \dots + \rho_{i_s} \leq \rho_j$$

for any $j \in \{2, \dots, k-1\}$ and $s \geq 2$ and $i_1, \dots, i_s \geq 1$ such that $i_1 + \dots + i_s = j$. In addition looking at the coefficient of

$$e_1 \wedge e_2 \wedge \dots \wedge e_{k-1} \wedge e_{i_1} \dots e_{i_s}$$

where $i_1 + \dots + i_s = k$, we find that

$$0 \leq \lambda_1 b/a + \lambda_{i_1} + \dots + \lambda_{i_s} - \lambda_k = \lambda_1(b/a + k(k+1)/2) - (\rho_{i_1} + \dots + \rho_{i_s} + \rho_2 + \dots + \rho_{k-1}),$$

and since $p_k^\infty \neq 0$ there is some i_1, \dots, i_s with $i_1 + \dots + i_s = k$ and

$$(22) \quad \lambda_1 b/a + \lambda_{i_1} + \dots + \lambda_{i_s} = \lambda_k$$

or equivalently

$$\lambda_1(b/a + k(k+1)/2) = \rho_{i_1} + \dots + \rho_{i_s} + \rho_2 + \dots + \rho_{k-1}.$$

Thus

$$(23) \quad \begin{aligned} p_k^\infty &= \lim_{t \rightarrow 0} e_1 \wedge (e_2 + t^{2\lambda_1 - \lambda_2} e_1^2) \wedge \dots \\ &\dots \wedge (e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1 + \dots + i_s = k-1} t^{\lambda_{i_1} + \dots + \lambda_{i_s} - \lambda_{k-1}} e_{i_1} \dots e_{i_s}) \wedge (t^{\lambda_1 b/a} \sum_{s=2}^k \sum_{i_1 + \dots + i_s = k} t^{\lambda_{i_1} + \dots + \lambda_{i_s} - r_k} e_{i_1} \dots e_{i_s}) \\ &= e_1 \wedge \dots \wedge (e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1 + \dots + i_s = k-1; \rho_{i_1} + \dots + \rho_{i_s} = \rho_{k-1}} e_{i_1} \dots e_{i_s}) \wedge \\ &\quad (\sum_{s=2}^k \sum_{\substack{i_1 + \dots + i_s = k \\ \lambda_1(b/a + k(k+1)/2) = \rho_{i_1} + \dots + \rho_{i_s} + \rho_2 + \dots + \rho_{k-1}}} e_{i_1} \dots e_{i_s}) \end{aligned}$$

is independent of e_k and hence is fixed by the automorphisms of \mathbb{C}^k which fix e_1, \dots, e_{k-1} and send e_k to $e_k + e_j$ for $j \in \{1, \dots, k-1\}$, as well as by the one-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^{\lambda_1} & 0 & \dots & 0 \\ 0 & t^{\lambda_2} & 0 & \dots & 0 \\ & & \dots & & \\ 0 & \dots & & 0 & t^{\lambda_k} \end{pmatrix}$$

of T_k . Thus to complete the proof of Lemma 6.5 and hence of Theorem 6.2, it suffices to find an additional one-dimensional stabiliser, which will be done in the rest of this section.

Letting

$$\mathbf{z} = [p_k] = [e_1 \wedge (e_2 + e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} e_{i_1} \dots e_{i_s})]$$

as at (18) we have

$$\begin{aligned} \lambda(t)\mathbf{z} &= [t^{\lambda_1} e_1 \wedge (t^{\lambda_2} e_2 + t^{2\lambda_1} e_1^2) \wedge \dots \wedge (\sum_{i_1+\dots+i_s=k} t^{\lambda_{i_1}+\dots+\lambda_{i_s}} e_{i_1} \dots e_{i_s})] = \\ &= [t^{\lambda_1+\dots+\lambda_k} (e_1 \wedge \dots \wedge e_k) + t^{\lambda_1+2\lambda_1+\lambda_3+\dots+\lambda_k} (e_1 \wedge e_1^2 \wedge e_3 \wedge \dots \wedge e_k) + \dots]. \end{aligned}$$

The generic term in this expression is

$$t^{\lambda_{\varepsilon_1}+\lambda_{\varepsilon_2}+\dots+\lambda_{\varepsilon_k}} (\mathbf{e}_{\varepsilon_1} \wedge \dots \wedge \mathbf{e}_{\varepsilon_k}), \quad \Sigma(\varepsilon_i) = i$$

where

$$(24) \quad \lambda_\tau = \sum_{i \in \tau} \lambda_i \text{ and } \mathbf{e}_\tau = \prod_{i \in \tau} e_i \text{ if } \tau = (i_1, \dots, i_s).$$

Definition 6.8. For any one-parameter subgroup λ as above let

- $m_\lambda = \min_{\substack{(\varepsilon_1, \dots, \varepsilon_k) \\ 1 \leq \Sigma(\varepsilon_i) \leq k}} (\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \dots + \lambda_{\varepsilon_k}),$
- $\mathbf{z}_\lambda = [\sum_{1 \leq \Sigma \varepsilon \leq k, \lambda_\varepsilon = m_\lambda} \mathbf{e}_\varepsilon],$
- $m_\lambda[i] = \min_{\Sigma(\varepsilon)=i} \lambda_\varepsilon$ for $1 \leq i \leq k,$
- $\mathbf{z}_\lambda[i] = [\sum_{\Sigma \varepsilon = i, \lambda_\varepsilon = m_\lambda[i]} \mathbf{e}_\varepsilon].$

Let \mathcal{O}_λ denote the $\mathrm{SL}(k)$ -orbit of \mathbf{z}_λ .

It is clear that the one-parameter subgroup $\tilde{\lambda}(t) = (t, t^2, \dots, t^k)$ stabilises \mathbf{z} , where \mathbf{z} is defined as at (18), and therefore $\mathbf{z} = \mathbf{z}_{\tilde{\lambda}}$ and its $\mathrm{SL}(k)$ -orbit is equal to its $\mathrm{GL}(k)$ -orbit.

We need a more precise description of the orbit structure of the closure of the orbit $\mathcal{O}_0 = \mathcal{O}_{\tilde{\lambda}}$. Since $\tilde{\lambda}_i = i\tilde{\lambda}_1$ for $i = 1, \dots, k$, for $\lambda \neq \tilde{\lambda}$ we have a smallest index $\sigma \in \{2, \dots, k\}$ with $\lambda_\sigma \neq \sigma\lambda_1$.

Definition 6.9. We call $\sigma = \mathrm{Head}(\lambda)$ the head of $\lambda = (\lambda_1, \dots, \lambda_n)$ if

$$\lambda_i = i\lambda_1 \text{ for } i < \sigma \text{ and } \lambda_\sigma \neq \sigma\lambda_1.$$

If $\lambda_\sigma < \sigma\lambda_1$ then we call λ regular ; otherwise we call λ degenerate.

We will say that a one-parameter subgroup λ is *maximal* if the closure of the orbit $\mathrm{GL}(k) \cdot \mathbf{z}_\lambda$ is a maximal boundary component of the closure of the orbit of \mathbf{z} .

Definition 6.10. Fix $0 < \varepsilon < 1$ and $2 \leq \sigma \leq k$. Let $\lambda^\sigma = (\lambda_1^\sigma, \dots, \lambda_k^\sigma)$ and $\mu^\sigma = (\mu_1^\sigma, \dots, \mu_k^\sigma)$ be the following one-parameter subgroups of $\mathrm{GL}(k)$:

$$(25) \quad \lambda_i^\sigma = i - \lfloor \frac{i}{\sigma} \rfloor \varepsilon \text{ for } 1 \leq i \leq k,$$

$$(26) \quad \mu_i^\sigma = \begin{cases} i & \text{for } i \neq \sigma, i \leq k, \\ \sigma + \varepsilon & \text{for } i = \sigma. \end{cases}$$

It is easy to see that $\text{Head}(\lambda^\sigma) = \text{Head}(\mu^\sigma) = \sigma$, and λ^σ is regular, whereas μ^σ is degenerate.

Definition 6.11. Let λ be a 1-parameter subgroup. We call

$$\#\{i : \mathbf{z}_\lambda[i] = e_i\}$$

the toral dimension of the limit point \mathbf{z}_λ .

Lemma 6.12. If the $\text{SL}(k)$ -orbit of p_k^∞ has codimension one in $\overline{\text{SL}(k)p_k}$, then $[p_k^\infty]$ lies in the orbit of one of $\mathbf{z}_{\lambda^2}, \dots, \mathbf{z}_{\lambda^k}$ or $\mathbf{z}_{\mu^2}, \dots, \mathbf{z}_{\mu^{k-1}}$.

Proof. We can assume that $[p_k^\infty] = \mathbf{z}_\lambda$ for some one-parameter subgroup λ . First suppose that λ is a regular one-parameter subgroup with $\text{Head}(\lambda) = \sigma$ and $[p_k^\infty] = \mathbf{z}_\lambda$. Without loss of generality we can assume that

$$\lambda_i = i \text{ for } i < \sigma \text{ and } \lambda_\sigma = \sigma - \varepsilon.$$

We will call $d(i) = \lfloor \frac{i}{\sigma} \rfloor$ the defect of i and $d(\tau) = d(i_1) + \dots + d(i_s)$ the defect of $\tau = (i_1, \dots, i_s)$, so that when $i \leq \sigma$ we have $d(i)\varepsilon = \rho_i$ as defined at (21). Since

$$\lambda_{(\underbrace{j, \sigma, \dots, \sigma}_m)} = j + m(\sigma - \varepsilon) \text{ for } 1 \leq j \leq \sigma - 1, m \geq 0,$$

we have

$$(27) \quad m_\lambda[i] \leq i - d(i)\varepsilon \text{ for } 1 \leq i \leq k.$$

If $\lambda_s < s - d(s)\varepsilon$ for $s > i$ and s is the smallest index with this property then $m_\lambda[s] = \lambda_s$ and $\mathbf{z}_\lambda[s] = e_s$, so

$$\mathbf{z}_\lambda[1] = e_1, \mathbf{z}_\lambda[\sigma] = e_\sigma, \mathbf{z}_\lambda[s] = e_s,$$

while \mathbf{z}_λ is independent of e_k by (23), so $[p_k^\infty]$ is fixed by a three-dimensional torus in $\text{SL}(k)$ and thus p_k^∞ is fixed by a two-dimensional torus in $\text{SL}(k)$ as well as a unipotent subgroup of dimension $k - 1$. So we can assume that $\lambda_i \geq i - d(i)\varepsilon$ for $1 \leq i \leq k$, and therefore

$$m_\lambda[i] = i - d(i)\varepsilon \text{ for } 1 \leq i \leq k.$$

So

$$(28) \quad \mathbf{e}_\tau \notin \mathbf{z}_\lambda[i] \text{ if } d(\tau) > d(i).$$

On the other hand the distinguished 1-parameter subgroup λ^σ is defined as $\lambda_i^\sigma = i - d(i)\varepsilon$, and therefore

$$(29) \quad \mathbf{z}_{\lambda^\sigma}[i] = \sum_{\Sigma(\tau)=i, d(\tau)=d(i)} \mathbf{e}_\tau.$$

Comparing (28) and (29) we conclude

$$\mathbf{z}_\lambda[i] \subset \mathbf{z}_{\lambda^\sigma}[i] \text{ for } 1 \leq i \leq k.$$

Now let μ be a degenerate 1-parameter subgroup with $\text{Head}(\mu) = \sigma$. Without loss of generality we can assume again that

$$\mu_i = i \text{ for } i < \sigma \text{ and } \mu_\sigma = \sigma + \varepsilon.$$

Since

$$\mu(\underbrace{1, \dots, 1}_i) = i \text{ for } 1 \leq i \leq k$$

we have

$$(30) \quad m_\mu[i] \leq i.$$

Again, $\mu_s < s$ cannot happen for $s > \sigma$ since in that case $\mathbf{z}_\mu[s] = e_s$ would hold and the codimension of $\text{SL}(k)p_k^\infty$ would be at least two. So $\mu_s \geq s$ and therefore $\mu_\tau \geq \Sigma(\tau)$ with strict inequality if $\sigma \in \tau$. Therefore

$$(31) \quad \mathbf{e}_\tau \notin \mathbf{z}_\mu[i] \text{ if } \sigma \in \tau.$$

On the other hand μ^σ satisfies equality in (30), and

$$(32) \quad \mathbf{z}_{\mu^\sigma}[i] = \sum_{\Sigma(\tau)=i, \sigma \notin \tau} \mathbf{e}_\tau.$$

Comparing (31) and (32) we get

$$\mathbf{z}_\mu[i] \subset \mathbf{z}_{\mu^\sigma}[i] \text{ for } 1 \leq i \leq k,$$

and so it remains to consider the possibility that $[p_k^\infty] = \mathbf{z}_{\mu^k}$. But by (22) there is some i_1, \dots, i_s with $i_1 + \dots + i_s = k$ and

$$\lambda_1 b/a + \lambda_{i_1} + \dots + \lambda_{i_s} = \lambda_k$$

and hence $\lambda_k > \lambda_{i_1} + \dots + \lambda_{i_s}$. Thus $[p_k^\infty]$ cannot be equal to \mathbf{z}_{μ^k} because the coefficient of $e_1 \wedge e_1^2 \dots \wedge e_1^k$ is nonzero for \mathbf{z}_{μ^k} but zero for $[p_k^\infty]$, and the result follows. \square

We summarize our information about the maximal boundary components in

Proposition 6.13. *We have $\mathbf{z}_{\lambda^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\lambda^\sigma}[i]$, where $\mathbf{z}_{\lambda^\sigma}[i] = \oplus_{\Sigma(\tau)=i, d(\tau)=d(i)} \mathbf{e}_\tau$, and $\mathbf{z}_{\mu^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\mu^\sigma}[i]$ where $\mathbf{z}_{\mu^\sigma}[i] = \oplus_{\Sigma(\tau)=i, \sigma \notin \tau} \mathbf{e}_\tau$.*

Remark 6.14. Since the one-parameter subgroup $\tilde{\lambda}(t) = (t, t^2, \dots, t^k)$ of $\text{GL}(k)$ stabilises $T_k \mathbf{z}$, it follows from Lemma 6.12 that it is enough to prove the codimension-at-least-two property we require only for the one-parameter subgroups $\tilde{\lambda}^\sigma$ (for $2 \leq s \leq k$) and $\tilde{\mu}^\sigma$ (for $2 \leq s \leq k-1$) of $\text{SL}(k)$ given by

$$\tilde{\lambda}^\sigma(t) = (\lambda^\sigma(t) \tilde{\lambda}(t)^{q_\sigma})^{n_\sigma}$$

and

$$\tilde{\mu}^\sigma(t) = (\mu^\sigma(t) \tilde{\lambda}(t)^{r_\sigma})^{m_\sigma}$$

for suitable $q_\sigma, r_\sigma \in \mathbb{Q}$ and $n_\sigma, m_\sigma \in \mathbb{Z}$. But we observed at (20) that the property is satisfied by a one-parameter subgroup λ of $\text{SL}(k)$ if $\lambda_{i_1} + \dots + \lambda_{i_s} < \lambda_j$ for any $j \in$

$\{2, \dots, k-1\}$ such that $i_1 + \dots + i_s = j$, so it is enough to consider the one-parameter subgroups $\tilde{\lambda}^\sigma$ for $2 \leq s \leq k$.

6.1. The limit of the stabilisers. In order to prove Lemma 6.5, it now suffices by Remark 6.14 to find a k -dimensional unipotent subgroup of the stabiliser $G_{\mathbf{z}_{\lambda^\sigma}}$ of $\mathbf{z}_{\lambda^\sigma}$ in $GL(k)$ for each σ when $\mathbf{z}_{\lambda^\sigma} = [p_k^\infty]$, since we know that p_k^∞ is fixed by a one-parameter subgroup of the maximal torus T_k of $SL(k)$, and any unipotent group which stabilises $\mathbf{z}_{\lambda^\sigma} = [p_k^\infty]$ also stabilises p_k^∞ .

In this subsection we will study the limits $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ of the stabiliser groups for the one-parameter subgroups λ^σ for $2 \leq \sigma \leq k$, and use this to prove Lemma 6.5, which together with Lemma 6.6 will complete the proof of Theorem 6.2.

Proposition 6.15. $G^\sigma = \lim_{t \rightarrow 0} G_{\lambda^\sigma(t)\mathbf{z}} \subset GL(k)$ is a k -dimensional subgroup of $G_{\mathbf{z}_{\lambda^\sigma}}$ which contains a $k-1$ -dimensional subgroup of the maximal unipotent subgroup U_k of $SL(k)$.

Proof. Consider the stabilizer

$$G_{\lambda^\sigma(t)\mathbf{z}} = \lambda^\sigma(t)^{-1} G_{\mathbf{z}} \lambda^\sigma(t).$$

Recall that

$$G_{\mathbf{z}} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{n-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^d \end{pmatrix} \right\}$$

where the polynomial in the (i, j) entry is

$$p_{i,j}(\alpha) = \sum_{a_1 + a_2 + \dots + a_i = j} \alpha_{a_1} \alpha_{a_2} \dots \alpha_{a_i}.$$

Therefore, the (i, j) entry of the stabilizer of $\lambda^\sigma(t)\mathbf{z}$ is

$$(33) \quad (G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma - \lambda_j^\sigma} p_{i,j}(\alpha)$$

If ε is small enough then $\lambda_1^\sigma < \lambda_2^\sigma < \dots < \lambda_k^\sigma$, and we define the positive number

$$(34) \quad n_i^\sigma = \max_{1 \leq j \leq n-i+1} (\lambda_{j+i-1}^\sigma - \lambda_j^\sigma), \quad i = 1, \dots, k.$$

Note that by definition $n_1^\sigma = 0$ for all σ .

Lemma 6.16. *Under the substitution*

$$\beta_i^\sigma = t^{-n_i^\sigma} \alpha_i^\sigma$$

we have

$$G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1, \dots, \beta_k) \in GL(\mathbb{C}[\beta_1, \dots, \beta_k][t]),$$

so the entries are polynomials in t with coefficients in $\mathbb{C}[\beta_1, \dots, \beta_k]$.

Proof. Compute the substitution as follows:

$$(35) \quad G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma - \lambda_j^\sigma} \sum_{a_1+a_2+\dots+a_i=j} \alpha_{a_1} \alpha_{a_2} \dots \alpha_{a_i} =$$

$$(36) \quad = \sum_{a_1+\dots+a_i=j} t^{\lambda_i^\sigma - \lambda_j^\sigma} t^{n_{a_1}^\sigma + n_{a_2}^\sigma + \dots + n_{a_i}^\sigma} \beta_{a_1} \beta_{a_2} \dots \beta_{a_i}.$$

By definition

$$n_{a_1}^\sigma \geq \lambda_{i+a_1-1}^\sigma - \lambda_i^\sigma; \quad n_{a_2}^\sigma \geq \lambda_{i+a_1+a_2-2}^\sigma - \lambda_{i+a_1-1}^\sigma; \quad \dots; \quad n_{a_j}^\sigma \geq \lambda_{i+a_1+\dots+a_i-i}^\sigma - \lambda_{i+a_1+\dots+a_{i-1}-(i-1)}^\sigma.$$

Adding up these inequalities and using $a_1 + \dots + a_i = j$ we get an alternating sum on the left cancelling up to

$$n_{a_1}^\sigma + \dots + n_{a_i}^\sigma \geq \lambda_j^\sigma - \lambda_i^\sigma.$$

Substituting this into (35) we get

$$(37) \quad (G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = \sum_{a_1+\dots+a_i=j} t^{\lambda_i^\sigma - \lambda_j^\sigma} t^{n_{a_1}^\sigma + n_{a_2}^\sigma + \dots + n_{a_i}^\sigma} \beta_{a_1} \beta_{a_2} \dots \beta_{a_i} \in \mathbb{C}[\beta_1, \dots, \beta_k][t].$$

This proves Lemma 6.16. □

As a corollary we get the existence of

$$G^\sigma = \lim_{t \rightarrow 0} G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1, \dots, \beta_k) \in GL(\mathbb{C}[\beta_1, \dots, \beta_k]).$$

To prove that $\dim G^\sigma = k$ and complete the proof of Proposition 6.15, for $1 \leq i \leq k$ choose $\theta(i)$ such that

$$(38) \quad n_i^\sigma = \lambda_{\theta(i)+i-1}^\sigma - \lambda_{\theta(i)}^\sigma$$

holds. Then

$$(39) \quad p_{\theta(i), \theta(i)+i-1}(\beta_1, \dots, \beta_k) = \sum_{a_1+\dots+a_{\theta(i)}=\theta(i)+i-1} t^{n_{a_1}^\sigma + \dots + n_{a_{\theta(i)}}^\sigma} \beta_{a_1} \dots \beta_{a_{\theta(i)}}$$

so

$$(40) \quad (G^\sigma)_{\theta(i), \theta(i)+i-1} = \lim_{t \rightarrow 0} t^{-n_i^\sigma} p_{\theta(i), \theta(i)+i-1}(\beta_1, \dots, \beta_k) = \lim_{t \rightarrow 0} (t^{n_i^\sigma} \beta_1^{\theta(i)-1} \beta_i + \dots) = \\ = \beta_1^{\theta(i)-1} \beta_i + q_{\theta(i), \theta(i)+i-1}$$

where

$$q_{\theta(i), \theta(i)+i-1} \in \mathbb{C}[\beta_1, \dots, \beta_k][t].$$

It follows that the elements $\frac{d}{dt} A^\sigma(t(e_1 + e_i)1) \in \text{Lie}(G^\sigma)$ are independent, where $t(e_1 + e_i) = (t, 0, \dots, 0, t, 0, \dots, 0)$ with the t 's are in the 1st and i th position if $i > 1$ but interpreted as $(2t, 0, \dots, 0)$ if $i = 1$. This completes the proof of Proposition 6.15. □

In order to prove Lemma 6.5, it now suffices to find an extra one-dimensional unipotent subgroup of the stabiliser $G_{\mathbf{z}_{\lambda^\sigma}}$ of $\mathbf{z}_{\lambda^\sigma}$ for each σ when $\mathbf{z}_{\lambda^\sigma} = [p_k^\infty]$, since we know that p_k^∞ is fixed by a one-parameter subgroup of the maximal torus T_k of $\mathrm{SL}(k)$ and by a $k - 1$ -dimensional unipotent subgroup of $G^\sigma = \lim_{\theta \rightarrow 0} G_{\lambda^\sigma(t)\mathbf{z}}$ which is contained in the standard maximal unipotent subgroup U_k of $\mathrm{SL}(k)$. It turns out that we have to distinguish three cases here.

Case 1: $\sigma = k$.

Proof. Let $T_\zeta \in \mathrm{GL}(k)$ denote the transformation

$$T_\zeta(e_i) = e_i \text{ for } i \neq k - 1 ; T_\zeta(e_{k-1}) = e_{k-1} + \zeta e_k \text{ for } \zeta \in \mathbb{C}.$$

Since e_{k-1} does not occur just in $\mathbf{z}_{\lambda^\sigma}[k - 1]$, T_ζ stabilises p_k^∞ . This gives us a subgroup of $\mathrm{SL}(k)$ of dimension at least $k + 1$ which stabilises p_k^∞ , because T_ζ is unipotent but not upper triangular if $\zeta \neq 0$. \square

Case 2: $\sigma < k$ and $k \not\equiv -1 \pmod{\sigma}$.

Proof. Let T be the transformation

$$(41) \quad T(e_i) = e_i \text{ for } i \neq k ; T(e_k) = e_k + \zeta e_\sigma.$$

Since e_k occurs only in $\mathbf{z}_{\lambda^\sigma}[k]$, and $\mathbf{z}_{\lambda^\sigma}[\sigma] = \sigma$, we have

$$(42) \quad \begin{aligned} T \cdot \mathbf{z}_{\lambda^\sigma} &= \mathbf{z}_{\lambda^\sigma}(e_1, \dots, e_{k-1}, e_k + \zeta e_\sigma) = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[\sigma - 1] \wedge e_\sigma \wedge \mathbf{z}_{\lambda^\sigma}[\sigma + 1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[k] + \\ &+ \zeta \cdot \mathbf{z}_{\lambda^\sigma}[1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[\sigma - 1] \wedge e_\sigma \wedge \mathbf{z}_{\lambda^\sigma}[\sigma + 1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[k - 1] \wedge e_\sigma = \mathbf{z}_{\lambda^\sigma}, \end{aligned}$$

so $T \in G_{\mathbf{z}_{\lambda^\sigma}}$.

It is slightly harder task to show that $T \notin G^\sigma = \lim_{\theta \rightarrow 0} G_{\lambda^\sigma(t)\mathbf{z}}$. First, we compute n_i for $i = k - \sigma$. We claim that for $k \not\equiv -1 \pmod{\sigma}$

$$(43) \quad n_{k-\sigma+1} = \lambda_k^\sigma - \lambda_\sigma^\sigma = \lambda_{k-\sigma+1}^\sigma - \lambda_1^\sigma.$$

Indeed,

$$\lambda_{j+k-\sigma-1} - \lambda_j = \dots \leq \lambda_k^\sigma - \lambda_\sigma^\sigma = \lambda_{k-\sigma+1}^\sigma - \lambda_1^\sigma$$

This means that we can choose $\theta(k - \sigma + 1) = \sigma$ in (38) and substitute into (40)

$$(44) \quad (G^\sigma)_{\sigma,k} = \beta_1^{\sigma-1} \beta_{k-\sigma+1} + q_{\sigma,k}(\beta_1, \dots, \beta_k),$$

where $q_{\sigma,k}(\beta_1, \dots, \beta_k)$ is a polynomial, whose monomials $\beta_{i_1}^{b_1} \dots \beta_{i_\sigma}^{b_\sigma}$ satisfy

$$(45) \quad i_1 b_1 + \dots + i_\sigma b_\sigma = k.$$

Moreover, we can also choose $\theta(k - \sigma + 1) = 1$, by (43), and then (40) gives us

$$(46) \quad (G^\sigma)_{1,k-\sigma+1} = \beta_{k-\sigma+1}.$$

Suppose now that $T \in G^\sigma$, that is

$$(47) \quad T = G^\sigma(\beta_1, \dots, \beta_k) \text{ for some } \beta_1 \in \mathbb{C}^*, \beta_2, \dots, \beta_k \in \mathbb{C}.$$

Let $(T)_{i,j}$ denote the (i, j) entry of T . Then

$$(T)_{\sigma,k} = \zeta, (T)_{i,j} = 0 \text{ for } i \neq j, (T)_{i,i} = 1.$$

Comparing the $(1, 1)$ and $(1, k - \sigma + 1)$ entries of T and G^σ we get

$$(48) \quad \beta_1 = 1, \beta_{k-\sigma+1} = 0.$$

Choose $\theta(i)$ for $i = 2, \dots, k$ as in (38) and let $\theta(k - \sigma + 1) = \sigma$. Since all off-diagonal entries of T but the (σ, k) are zero, (47) forces the following equations

$$(49) \quad \beta_i + q_{\theta(i), \theta(i)+i-1} = 0 \text{ for } i \neq k - \sigma + 1,$$

$$(50) \quad \beta_{k-\sigma+1} + q_{\sigma,k} = \zeta.$$

By (48), these are $k - 1$ polynomial equations in $k - 2$ variables, and the Jacobian at 0 is the origin, so we have finitely many solutions near the origin. Therefore, for some ζ , it follows that T is not in G^σ . \square

Case 3: $\sigma < k$ and $d = -1 \pmod{\sigma}$.

Proof. This case works very similarly to the previous one. Suppose $k - 1 > \sigma$, that is, if $k = c\sigma - 1$ where $c \geq 2$ (this holds because $k \geq \sigma$), the condition is that $c\sigma - 2 > \sigma$, which is true for all $k \geq 4$.

Let T be the transformation

$$(51) \quad T(e_i) = e_i \text{ for } i \neq k, k - 1; T(e_{k-1}) = e_{k-1} + \zeta e_\sigma; T(e_k) = e_k + \zeta e_\sigma$$

First we check again that $T \in G_{\mathbf{z}_{\lambda^\sigma}}$. We have

$$\begin{aligned} \mathbf{z}_{\lambda^\sigma}[\sigma] &= e_\sigma; \\ \mathbf{z}_{\lambda^\sigma}[\sigma + 1] &= e_{\sigma+1} + e_1 e_\sigma; \\ \mathbf{z}_{\lambda^\sigma}[k] &= e_k + \sum_{i=1}^{k-1} e_i e_{k-i}. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} (52) \quad T \cdot \mathbf{z}_{\lambda^\sigma} &= \mathbf{z}_{\lambda^\sigma}(e_1, \dots, e_{k-2}, e_{k-1} + \zeta e_\sigma, e_k + \zeta e_{\sigma+1}) = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[k-2] \wedge (\mathbf{z}_{\lambda^\sigma}[k-1] + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma]) \wedge (\mathbf{z}_{\lambda^\sigma}[k] + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma+1]) = \\ &= \mathbf{z}_{\lambda^\sigma}[1] \wedge \dots \wedge \mathbf{z}_{\lambda^\sigma}[k] = \mathbf{z}_{\lambda^\sigma}. \end{aligned}$$

Now we prove that $T \notin G^\sigma$ in a similar way to the second case above. Since $k - 1 \neq -1 \pmod{\sigma}$ we can substitute $k - 1$ instead of k in (43):

$$(53) \quad n_{k-\sigma} = \lambda_{k-1}^\sigma - \lambda_\sigma^\sigma = \lambda_{k-\sigma}^\sigma - \lambda_1^\sigma.$$

Moreover, we also get the extra equation

$$(54) \quad n_{k-\sigma} = \lambda_k^\sigma - \lambda_{\sigma+1}^\sigma,$$

and similarly to (44) and (46) it follows that

$$(55) \quad (G^\sigma)_{\sigma,k-1} = \beta_1^{\sigma-1} \beta_{k-\sigma} + q_{\sigma,k-1}(\beta_1, \dots, \beta_k);$$

$$(56) \quad (G^\sigma)_{\sigma+1,k} = \beta_1^\sigma \beta_{k-\sigma} + q_{\sigma+1,k}(\beta_1, \dots, \beta_k);$$

$$(57) \quad (G^\sigma)_{1,k-\sigma} = \beta_{k-\sigma}.$$

Since T differs from the identity matrix only by the entries

$$(T)_{\sigma,k-1} = (T)_{\sigma+1,k} = \zeta,$$

the equality

$$T = G^\sigma(\beta_1, \dots, \beta_k)$$

forces $\beta_{k-\sigma} = 0, \beta_1 = 1$ and the analogue of (49), (50):

$$(58) \quad \beta_i + q_{\theta(i), \theta(i)+1} = 0 \text{ for } i \neq k - \sigma$$

$$(59) \quad \beta_{k-\sigma} + q_{\sigma,k-1} = \zeta$$

$$(60) \quad \beta_{k-\sigma} + q_{\sigma+1,k} = \zeta$$

which are, again, $k+1$ nondegenerate polynomial equations in $k-1$ variables, such that for some ζ there is no solution. \square

We have now proved Lemma 6.5, which together with Lemma 6.6 completes the proof of Theorem 6.2.

7. GEOMETRIC DESCRIPTION OF DEMAILLY-SEMPLÉ INVARIANTS

As an immediate consequence of Corollary 6.3, we can now prove Theorem 3.3 in the case when $p = 1$.

Theorem 7.1. *If $k \geq 2$ then $\mathbb{G}'_k = \mathbb{U}_k$ is a Grosshans subgroup of the special linear group $SL(k)$, so that $O(SL(k)^{\mathbb{U}_k})^{SL(k)}$ is a finitely generated complex algebra and moreover every linear action of \mathbb{U}_k or \mathbb{G}_k on an affine or projective variety Y (with respect to an ample linearisation) which extends to a linear action of $GL(k)$ has finitely generated invariants.*

In particular we have the special case of Theorem 3.2 when $p = 1$.

Theorem 7.2. *The fibre $O((J_k)_x)^{\mathbb{U}_k}$ of the bundle E_k^n is a finitely generated graded complex algebra.*

Proof. We have

$$O((J_k)_x)^{\mathbb{U}_k} \cong (O((J_k)_x) \otimes O(SL(k)^{\mathbb{U}_k})^{SL(k)})$$

which is finitely generated because $O(SL(k)^{\mathbb{U}_k})^{SL(k)}$ is finitely generated and $SL(k)$ is reductive. \square

Theorem 6.2 also allows us to describe the algebra $\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k}$. In §6 we constructed an embedding of $\mathrm{SL}(k)/\mathbb{U}_k$ in the affine space $\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$ for suitable large K , and in Theorem 6.2 we proved that the boundary components of the closure $\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ of its image have codimension at least two. Thus we obtain the following corollary of Theorem 6.2:

Theorem 7.3. (i) *If $k \geq 4$ then the canonical affine completion*

$$\mathrm{SL}(k)/\mathbb{U}_k = \mathrm{Spec}(\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k})$$

of $\mathrm{SL}(k)/\mathbb{U}_k$ is isomorphic to the closure $\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ of the orbit $\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K}) \cong \mathrm{SL}(k)/\mathbb{U}_k$ of $p_k \otimes e_1^{\otimes K}$ in $\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$ where $K = M(1 + 2 + \cdots + k) + 1$ for any strictly positive integer M ;

(ii) *The algebra*

$$\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k}$$

is generated by the Plücker coordinates on $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k))$, which can be expressed as

$$\{\Delta_{\mathbf{i}_1, \dots, \mathbf{i}_s} : s \leq k\},$$

where \mathbf{i}_j denotes a multi-index corresponding to basis elements of $\mathrm{Sym}^{\leq k}(\mathbb{C}^k)$, and $\Delta_{\mathbf{i}_1, \dots, \mathbf{i}_s}$ is the corresponding minor of $\phi(f' \dots, f^{(k)}) \in \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^{\leq k}(\mathbb{C}^k))$, together with the coordinates f'_1, \dots, f'_k of f' .

It follows immediately from this theorem that the non-reductive GIT quotient

$$(J_k)_x/\mathbb{U}_k = \mathrm{Spec}(\mathcal{O}((J_k)_x)^{\mathbb{U}_k})$$

is isomorphic to the reductive GIT quotient

$$((J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})})/\mathrm{SL}(k).$$

This can be identified with the quotient of the open subset $((J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})})^{ss}$ of $\mathrm{SL}(k)$ -semistable points of $(J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ by the equivalence relation \sim such that $y \sim z$ if and only if the closures of the $\mathrm{SL}(k)$ -orbits of y and z intersect in $((J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})})^{ss}$. Equivalently it can be identified with the closed $\mathrm{SL}(k)$ -orbits in $((J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})})^{ss}$. Since $\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})$ is the union of finitely many $\mathrm{SL}(k)$ -orbits, with stabilisers $H_1 = \mathbb{U}_k, H_2, \dots, H_s$, say, we can stratify $(J_k)_x/\mathbb{U}_k$ so that the stratum corresponding to H_j is identified with the H_j -orbits in $(J_k)_x$ such that the corresponding $\mathrm{SL}(k)$ -orbit in $(J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ is semistable and closed in $((J_k)_x \times \overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})})^{ss}$.

Example 7.4. *When $k = 2$ we have*

$$J_2^{\mathrm{reg}}(1, 2) = \{(f'_1, f'_2, f''_1, f''_2) \in (\mathbb{C}^2)^2; (f'_1, f'_2) \neq (0, 0)\},$$

and fixing a basis $\{e_1, e_2\}$ of \mathbb{C}^2 and the induced basis $\{e_1, e_2, e_1^2, e_1e_2, e_2^2\}$ of $\mathbb{C}^2 \oplus \text{Sym}^2 \mathbb{C}^2$, the map $\phi : J_2(1, 2) = \text{Hom}(\mathbb{C}^2, \mathbb{C}^2) \rightarrow \text{Hom}(\mathbb{C}^2, \text{Sym}^{\leq 2} \mathbb{C}^2)$ of (14) is given by

$$(f'_1, f'_2, f''_1, f''_2) \mapsto \begin{pmatrix} f'_1 & f'_2 & 0 & 0 & 0 \\ \frac{1}{2!}f''_1 & \frac{1}{2!}f''_2 & (f'_1)^2 & f'_1f'_2 & (f'_2)^2 \end{pmatrix}.$$

The 2×2 minors of this 2×5 matrix are $(f'_1)^3$, $(f'_1)^2f'_2$, $f'_1(f'_2)^2$, $(f'_2)^3$ and

$$\Delta_{[1,2]} = f'_1f''_2 - f''_1f'_2.$$

On $SL(2)$ we have $\Delta_{[1,2]} = 1$ and the algebra of invariants $\mathcal{O}(SL(2))^{\mathbb{U}_2}$ is generated by f'_1 and f'_2 , as expected since $SL(2)/\mathbb{U}_2 \cong \mathbb{C}^2 \setminus \{0\}$ and its canonical affine completion $SL(2)/\mathbb{U}_2$ is \mathbb{C}^2 .

Example 7.5. When $k = 3$ the finite generation of the Demailly-Semple algebra $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ was proved by Rousseau in [27]. We have

$$J_3^{\text{reg}}(1, 3) = \{(f'_1, f'_2, f'_3, f''_1, f''_2, f''_3, f'''_1, f'''_2, f'''_3) \in (\mathbb{C}^3)^3; (f'_1, f'_2, f'_3) \neq (0, 0, 0)\},$$

and if we fix a basis $\{e_1, e_2, e_3\}$ of \mathbb{C}^3 and the induced basis

$$\{e_1, e_2, e_3, e_1^2, e_1e_2, e_2^2, e_1e_3, e_2e_3, e_3^2, e_1^3, e_1^2e_2, \dots, e_3^3\}$$

of $\mathbb{C}^3 \oplus \text{Sym}^2 \mathbb{C}^3 \oplus \text{Sym}^3 \mathbb{C}^3$, the map $\phi : \text{Hom}(\mathbb{C}^3, \mathbb{C}^3) \rightarrow \text{Hom}(\mathbb{C}^3, \text{Sym}^{\leq 3} \mathbb{C}^3)$ in (14) sends

$$(f'_1, f'_2, f'_3, f''_1, f''_2, f''_3, f'''_1, f'''_2, f'''_3)$$

to a 3×19 matrix, whose first 9 columns (corresponding to $\text{Sym}^{\leq 2} \mathbb{C}^3$) are

$$\begin{pmatrix} f'_1 & f'_2 & f'_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2!}f''_1 & \frac{1}{2!}f''_2 & \frac{1}{2!}f''_3 & (f'_1)^2 & f'_1f'_2 & (f'_2)^2 & f'_1f'_3 & f'_2f'_3 & (f'_3)^2 \\ \frac{1}{3!}f'''_1 & \frac{1}{3!}f'''_2 & \frac{1}{3!}f'''_3 & f'_1f''_1 & f'_1f''_2 + f''_1f'_2 & f'_2f''_2 & f'_1f''_3 + f'_3f''_1 & f'_2f''_3 + f'_3f''_2 & f'_3f''_3 \end{pmatrix},$$

and the remaining 10 columns (corresponding to $\text{Sym}^3 \mathbb{C}^3$) are

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (f'_1)^3 & (f'_1)^2f'_2 & f'_1(f'_2)^2 & (f'_2)^3 & f'_1(f'_3)^2 & (f'_1)^2f'_3 & (f'_2)^2f'_3 & f'_2(f'_3)^2 & (f'_3)^3 & f'_1f'_2f'_3 \end{pmatrix}.$$

The 3×3 minors of this matrix together with f'_1, f'_2, f'_3 generate the algebra of invariants $\mathcal{O}(SL(3))^{\mathbb{U}_3}$.

8. GENERALIZED DEMAILLY-SEMPLÉ JET BUNDLES

The aim of this section is to extend the earlier constructions for $p = 1$ to generalized Demailly-Semple invariant jet differentials when $p > 1$.

Let X be a compact, complex manifold of dimension n . We fix a parameter $1 \leq p \leq n$, and study the maps $\mathbb{C}^p \rightarrow X$. Recall that as before we fix the degree k of the map, and introduce the bundle $J_{k,p} \rightarrow X$ of k -jets of maps $\mathbb{C}^p \rightarrow X$, so that the fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}^p, 0) \rightarrow (X, x)$, with the equivalence relation $f \sim g$ if and only if all derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for

$0 \leq j \leq k$. Recall also that $\mathbb{G}_{k,p}$ is the group of k -jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$, which has a natural fibrewise right action on $J_{k,p}$ with the matrix representation given by

$$(61) \quad G_{k,p} = \begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \dots & \\ 0 & 0 & \Phi_1^3 & \dots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \Phi_1^k \end{pmatrix},$$

for $G_{k,p} \in \mathbb{G}_{p,k}$ where $\Phi_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$ and $\det \Phi_1 \neq 0$. Recall also that $\mathbb{G}_{k,p}$ is generated along its first p rows, in the sense that the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The parameters in the $(1, m)$ block are indexed by a basis of $\text{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p$, so they are of the form α_ν^l where $\nu \in \binom{p+m-1}{m-1}$ is an m -tuple and $1 \leq l \leq p$, and the polynomial in the (l, m) block and entry indexed by $\tau = (\tau[1], \dots, \tau[l]) \in \binom{p+l-1}{l-1}$ and $\nu \in \binom{p+m-1}{m-1}$ is given by

$$(62) \quad (G_{k,p})_{\tau,\nu} = \sum_{\nu_1 + \dots + \nu_l = \nu} \alpha_{\nu_1}^{\tau[1]} \alpha_{\nu_2}^{\tau[2]} \dots \alpha_{\nu_l}^{\tau[l]}.$$

Recall also that $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \rtimes \text{GL}(p)$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ by $\text{GL}(p)$, and that the generalized Demailly-Semple jet bundle $E_{k,p,m} \rightarrow X$ of invariant jet differentials of order k and weighted degree (m, \dots, m) consists of the jet differentials which transform under any reparametrization $\phi \in \mathbb{G}_{k,p}$ of $(\mathbb{C}^p, 0)$ as

$$Q(f \circ \phi) = (J_\phi)^m Q(f) \circ \phi,$$

where $J_\phi = \det \Phi_1$ denotes the Jacobian of ϕ , so that $E_{k,p} = \oplus_{m \geq 0} E_{k,p,m}$ is the graded algebra of $\mathbb{G}'_{k,p}$ -invariants where $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes \text{SL}(p)$.

8.1. Geometric description for $p > 1$. As in the case when $p = 1$ our goal is to prove that $\mathbb{G}'_{k,p}$ is a Grosshans subgroup of $\text{SL}(\text{sym}^{\leq k}(p))$ where $\text{sym}^{\leq k}(p) = \sum_{i=1}^k \dim \text{Sym}^i \mathbb{C}^p$ by finding a suitable embedding of the quotient $\text{SL}(\text{sym}^{\leq k}(p))/\mathbb{G}'_{k,p}$.

Remark 8.1. In [25] Pacienza and Rousseau generalize the inductive process given in [5] of constructing a smooth compactification of the Demailly-Semple jet bundles. Using the concept of a directed manifold, they define a bundle $X_{k,p} \rightarrow X$ with smooth fibres, and the effective locus $Z_{k,p} \subset X_{k,p}$, and a holomorphic embedding $J_{k,p}^{\text{reg}}/\mathbb{G}_{k,p} \hookrightarrow Z_{k,p}$ which identifies $J_{k,p}^{\text{reg}}/\mathbb{G}_{k,p}$ with $Z_{k,p}^{\text{reg}} = X_{k,p}^{\text{reg}} \cap Z_{k,p}$, so that $Z_{k,p}$ is a relative compactification of $J_{k,p}/\mathbb{G}_{k,p}$. We choose a different approach, generalizing the test curve model, resulting in a holomorphic embedding of $J_{k,p}/\mathbb{G}_{k,p}$ into a partial flag manifold and a different compactification, which is a singular subvariety of the partial flag manifold, such that the invariant jet differentials of degree divisible by $\text{sym}^{\leq k}(p)$ are given by polynomial expressions in the Plücker coordinates.

Fix $x \in X$ and an identification of $T_x X$ with \mathbb{C}^n ; then let $J_k(p, n) = J_{k,p,x}$ as defined in §2. Let

$$J_k^{\text{reg}}(p, n) = \{\gamma \in J_k(p, n) : \Gamma_1 \text{ is non-degenerate}\}$$

where γ is represented by

$$\mathbf{u} \mapsto \gamma(\mathbf{u}) = \Gamma_1 \mathbf{u} + \Gamma_2 \mathbf{u}^2 + \dots + \Gamma_k \mathbf{u}^k$$

with $\Gamma_i \in \text{Hom}(\text{Sym}^i \mathbb{C}^p, \mathbb{C}^p)$. Let $N \geq n$ be any integer and define

$$\Upsilon_{k,p} = \{\Psi \in J_k(n, N) : \exists \gamma \in J_k^{\text{reg}}(p, n) : \Psi \circ \gamma = 0\}.$$

Remark 8.2. The global singularity theory description of $\Upsilon_{k,p}$ is

$$\Upsilon_{k,p} \doteq \{p = (p_1, \dots, p_N) \in J_k(n, N) : \mathbb{C}[z_1, \dots, z_n] / \langle p_1, \dots, p_N \rangle \cong \mathbb{C}[x, y] / \langle z_1, \dots, z_n \rangle^{k+1}\}.$$

Note, again, as in the $p = 1$ case, that if $\gamma \in J_k^{\text{reg}}(p, n)$ is a test surface of $\Psi \in \Upsilon_{k,p}$, and $\varphi \in \mathbb{G}_k$ is a holomorphic reparametrization of \mathbb{C}^p , then $\gamma \circ \varphi$ is, again, a test surface of Ψ :

$$(63) \quad \begin{array}{ccccccc} \mathbb{C}^p & \xrightarrow{\varphi} & \mathbb{C}^p & \xrightarrow{\gamma} & \mathbb{C}^n & \xrightarrow{\Psi} & \mathbb{C}^N \\ & & \Psi \circ \gamma = 0 & \Rightarrow & \Psi \circ (\gamma \circ \varphi) = 0 & & \end{array}$$

Example 8.3. Let $k = 2, p = 2$ and let $\Psi(\mathbf{z}) = \Psi' \mathbf{z} + \Psi'' \mathbf{z}^2$ for $\mathbf{z} \in \mathbb{C}^n$, and

$$\gamma(u_1, u_2) = \gamma_{10}u_1 + \gamma_{01}u_2 + \gamma_{20}u_1^2 + \gamma_{11}u_1u_2 + \gamma_{02}u_2^2, \quad \gamma_{ij} \in \mathbb{C}^n.$$

Then $\Psi \circ \gamma = 0$ has the form

$$(64) \quad \begin{aligned} \Psi'(\gamma_{10}) &= 0; \quad \Psi'(\gamma_{01}) = 0 \\ \Psi'(\gamma_{20}) + \Psi''(\gamma_{10}, \gamma_{10}) &= 0, \quad ; \quad \Psi'(\gamma_{11}) + 2\Psi''(\gamma_{10}, \gamma_{01}) = 0, \quad ; \quad \Psi'(\gamma_{01}) + \Psi''(\gamma_{01}, \gamma_{01}) = 0, \end{aligned}$$

We introduce

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n, N) : \Psi \circ \gamma = 0\}$$

and the following analogue of $J_k^o(1, n)$:

$$J_k^o(n, N) = \{\Psi \in J_k(n, N) : \dim \ker \Psi = p\}.$$

The proof of the following proposition is analogous to that of Proposition 4.7 in [2], and we omit the details. We use the notation

$$\text{sym}^i(p) = \dim(\text{Sym}^i \mathbb{C}^p); \quad \text{sym}^{\leq k}(p) = \dim(\mathbb{C}^p \oplus \text{Sym}^2 \mathbb{C}^p \oplus \dots \oplus \text{Sym}^k \mathbb{C}^p) = \sum_{i=1}^k \text{sym}^i p.$$

Proposition 8.4. (i) If $\gamma \in J_k^{\text{reg}}(p, n)$ then $\mathcal{S}_\gamma \subset J_k(n, N)$ is a linear subspace of codimension $N \text{sym}^{\leq k}(p)$.
(ii) For any $\gamma \in J_k^{\text{reg}}(p, n)$, the subset $\mathcal{S}_\gamma \cap J_k^o(n, N)$ of \mathcal{S}_γ is dense.

- (iii) If $\Psi \in J_k^o(n, N)$, then Ψ belongs to at most one of the spaces \mathcal{S}_γ . More precisely, if $\gamma_1, \gamma_2 \in J_k^{\text{reg}}(p, n)$, $\Psi \in J_k^o(n, N)$ and $\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0$, then there exists $\varphi \in J_k^{\text{reg}}(p, p)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.
- (iv) Given $\gamma_1, \gamma_2 \in J_k^{\text{reg}}(1, n)$, we have $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ if and only if there is some $\varphi \in J_k^{\text{reg}}(1, 1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

With the notation

$$\Upsilon_{k,p} = \Upsilon_{k,p} \cap J_k^o(n, N),$$

we deduce from Proposition 8.4 the following

Corollary 8.5. $\Upsilon_{k,p}^0$ is a dense subset of $\Upsilon_{k,p}$, and $\Upsilon_{k,p}^0$ has a fibration over the orbit space $J_k^{\text{reg}}(p, n)/J_k^{\text{reg}}(p, p) = J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p}$ with linear fibres.

Remark 8.6. In fact, Proposition 8.4 says a bit more, namely that $\Upsilon_{k,p}^0$ is fibrewise dense in $\Upsilon_{k,p}$ over $J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p}$, but we will not use this stronger statement.

By the first part of Proposition 8.4 the assignment $\gamma \rightarrow \mathcal{S}_\gamma$ defines a map

$$\nu : J_k^{\text{reg}}(p, n) \rightarrow \text{Grass}(kN, J_k(n, N))$$

which, by the fourth part, descends to the quotient

$$(65) \quad \bar{\nu} : J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \text{Grass}(kN, J_k(n, N))$$

(cf. Proposition 4.4). Next, we want to rewrite this embedding in terms of the identifications introduced in §5. So we

- identify $J_k(p, n)$ with $\text{Hom}(\mathbb{C}^{\text{sym}^1 p} \oplus \dots \oplus \mathbb{C}^{\text{sym}^k p}, \mathbb{C}^n) = \text{Hom}(\mathbb{C}^{\text{sym}^{\leq k} p}, \mathbb{C}^n)$ where $\text{sym}^j p = \dim \text{Sym}^j \mathbb{C}^p$ and $\text{sym}^{\leq k} p = \sum_{j=1}^k \text{sym}^j p$;
- identify $J_k(n, 1)^*$ with $\text{Sym}^{\leq k} \mathbb{C}^n = \bigoplus_{l=1}^k \text{Sym}^l \mathbb{C}^n$.

We think of an element ν of $\text{Hom}(\mathbb{C}^{\text{sym}^{\leq k} p}, \mathbb{C}^n)$ as an $n \times \text{sym}^{\leq k} p$ matrix, with column vectors in \mathbb{C}^n . These columns correspond to basis elements of $\mathbb{C}^{\text{sym}^1 p} \oplus \dots \oplus \mathbb{C}^{\text{sym}^k p}$, and the columns in the i th component are indexed by i -tuples $1 \leq t_1 \leq t_2 \leq \dots \leq t_i \leq p$, or equivalently by

$$(e_{t_1} + e_{t_2} + \dots + e_{t_i}) \in \mathbb{Z}_{\geq 0}^p$$

where $e_j = (0, \dots, 1, \dots, 0)$ with 1 in the j th place, giving us

$$\nu = (\nu_{10\dots 0}, \nu_{01\dots 0}, \dots, \nu_{0\dots 0k}) \in \text{Hom}(\mathbb{C}^{\text{sym}^{\leq k} p}, \mathbb{C}^n).$$

The elements of $J_k^{\text{reg}}(p, n)$ correspond to matrices whose first p columns are linearly independent. When $n \geq \text{sym}^{\leq k} p$ there is a smaller dense open subset $J_k^{\text{nondeg}}(p, n) \subset J_k^{\text{reg}}(p, n)$ consisting of the $n \times \text{sym}^{\leq k} p$ matrices of rank $\text{sym}^{\leq k} p$.

Define the following map, whose components correspond to the equations in (64):

$$(66) \quad \phi : \text{Hom}(\mathbb{C}^{\text{sym}^{\leq k} p}, \mathbb{C}^n) \rightarrow \text{Hom}(\mathbb{C}^{\text{sym}^{\leq k} p}, \text{Sym}^{\leq k} \mathbb{C}^n)$$

$$(\nu_{10\dots 0}, \nu_{01\dots 0}, \dots, \nu_{0\dots 0k}) \mapsto (\dots, \sum_{\mathbf{s}_1 + \mathbf{s}_2 + \dots + \mathbf{s}_j = \mathbf{s}} \nu_{\mathbf{s}_1} \nu_{\mathbf{s}_2} \dots \nu_{\mathbf{s}_j}, \dots),$$

where on the right hand side $\mathbf{s} \in \mathbb{Z}_{\geq 0}^p$.

Example 8.7. If $k = p = 2$ then ϕ is given by

$$\phi(v_{10}, v_{01}, v_{20}, v_{11}, v_{02}) = (v_{10}, v_{01}, v_{20} + v_{10}^2, v_{11} + 2v_{10}v_{01}, v_{02} + v_{01}^2).$$

Let $P_{k,p} \subset \mathrm{GL}_{\mathrm{sym}^{\leq k}(p)}$ denote the standard parabolic subgroup with Levi subgroup

$$\mathrm{GL}(\mathrm{sym}^1 p) \times \dots \times \mathrm{GL}(\mathrm{sym}^k p),$$

where $\mathrm{sym}^j p = \dim \mathrm{Sym}^j \mathbb{C}^p$ and $\mathrm{sym}^{\leq k}(p) = \sum_{j=1}^k \mathrm{sym}^j p$. Then (65) has the following reformulation, analogous to Proposition 5.1.

Proposition 8.8. The map ϕ in (66) is a $\mathbb{G}_{k,p}$ -invariant algebraic morphism

$$\phi : J_k^{\mathrm{reg}}(p, n) \rightarrow \mathrm{Hom}(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k} \mathbb{C}^n)$$

which induces an injective map ϕ^{Grass} on the $\mathbb{G}_{k,p}$ -orbits:

$$\phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Grass}_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n)$$

and

$$\phi^{\mathrm{Flag}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Flag}_{\mathrm{sym}^1(p), \dots, \mathrm{sym}^k(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n) \hookrightarrow \mathrm{Hom}(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k} \mathbb{C}^n)/P_{k,p}.$$

Composition with the Plücker embedding gives

$$\phi^{\mathrm{Proj}} = \mathrm{Pluck} \circ \phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p, n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)} \mathrm{Sym}^{\leq k} \mathbb{C}^n).$$

As in the case when $p = 1$, we introduce the following notation

$$X_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{reg}}(p, n)), \quad Y_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(p, n)) \subset \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}}(\mathrm{Sym}^{\leq k} \mathbb{C}^n)).$$

Definition 8.9. Let $n \geq \mathrm{sym}^{\leq k}(p) = \mathrm{sym}^1(p) + \dots + \mathrm{sym}^k(p)$. Then the open subset of $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n))$ where the projection to $\wedge^{\mathrm{sym}^{\leq k}(p)} \mathbb{C}^n$ is nonzero is denoted by $A_{n,k,p}$.

Since ϕ^{Grass} and ϕ^{Proj} are $\mathrm{GL}(n)$ -equivariant, and for $n \geq \mathrm{sym}^{\leq k}(p)$ the action of $\mathrm{GL}(n)$ is transitive on $\mathrm{Hom}^{\mathrm{nondeg}}(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n)$, we have

Lemma 8.10. (i) If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ is the $\mathrm{GL}(n)$ orbit of

$$(67) \quad \mathbf{z} = \phi^{\mathrm{Proj}}(e_1, \dots, e_{\mathrm{Sym}^{\leq k}(p)}) = [\wedge_{j_1 + \dots + j_p \leq k} \sum_{\mathbf{i}_1 + \dots + \mathbf{i}_s = (j_1, \dots, j_p)} e_{\mathbf{i}_1} \dots e_{\mathbf{i}_s}]$$

in $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n))$.

(ii) If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ and $Y_{n,k,p}$ are finite unions of $\mathrm{GL}(n)$ orbits.

(iii) For $k > n$ the images $X_{n,k,p}$ and $Y_{n,k,p}$ are $\mathrm{GL}(n)$ -invariant quasi-projective varieties, though they have no dense $\mathrm{GL}(n)$ orbit.

Similar statements hold for the closure of the image in the Grassmannian

$$\mathrm{Grass}_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n)$$

(or equivalently in the projective space $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k} \mathbb{C}^n))$).

Lemma 8.11. Let $n \geq \mathrm{sym}^{\leq k}(\mathbb{C}^n)$; then

- (i) $A_{n,k,p}$ is invariant under the $GL(n)$ action on $\mathbb{P}(\wedge^{\text{sym}^{\leq k}(p)}(\text{Sym}^{\leq k}\mathbb{C}^n))$;
- (ii) $X_{n,k,p} \subset A_{n,k,p}$, although $Y_{n,k,p} \not\subset A_{n,k,p}$;
- (iii) $\overline{X}_{n,k,p}$ is the union of finitely many $GL(n)$ -orbits.

9. AFFINE EMBEDDINGS OF $SL(\text{sym}^{\leq k}p)/\mathbb{G}_{k,p}$

In this section we study the case when $n = \text{sym}^{\leq k}p$ and so $GL(n) \subset J_k^{\text{reg}}(p, n)$. In the previous section we embedded $J_k^{\text{reg}}(p, n)/\mathbb{G}_{k,p}$ in the affine space $A_{n,k,p} \subset \mathbb{P}(\wedge^n \text{Sym}^{\leq k}\mathbb{C}^n)$, which can be restricted to $GL(n)$ to give us an embedding

$$GL(n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^n \text{Sym}^{\leq k}\mathbb{C}^n)$$

as the $GL(n)$ orbit of

$$[\dots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\dots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \dots e_{\mathbf{s}_j} \wedge \dots].$$

Equivalently we have $SL(n)/(SL(n) \cap \mathbb{G}_{k,p}) = SL(n)/\mathbb{G}'_{k,p} \rtimes F_{k,p}$ embedded in $\wedge^k(\text{Sym}^{\leq k}\mathbb{C}^n)$ as the $SL(n)$ orbit of

$$p_{k,p} = \dots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\dots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \dots e_{\mathbf{s}_j} \wedge \dots,$$

where $SL(n) \cap \mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{G}'_{k,p} \rtimes F_{k,p}$ of $\mathbb{G}'_{k,p}$ by the finite group $F_{k,p}$ of $l_{k,p}$ th roots of unity in \mathbb{C} for $l_{k,p} = \sum_{i=1}^k i \text{sym}^i p$. In analogy with §6 we can consider an embedding of $SL(n)/\mathbb{G}'_{k,p}$ in

$$\wedge^n(\text{Sym}^{\leq k}\mathbb{C}^n) \otimes (\wedge^p(\mathbb{C}^n))^{\otimes K}$$

for suitable K and its closure in this affine space. We expect the following result generalising Theorem 6.2.

Conjecture 9.1. *Let $K = M(\sum_{i=1}^k i \text{sym}^i p) + 1$ where $M \in \mathbb{N}$. Then the point*

$$p_{k,p} \otimes (e_1 \wedge \dots \wedge e_p)^{\otimes K} \in \wedge^n(\text{Sym}^{\leq k}\mathbb{C}^n) \otimes (\wedge^p(\mathbb{C}^n))^{\otimes K}$$

where

$$p_{k,p} = \dots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\dots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \dots e_{\mathbf{s}_j} \wedge \dots$$

has stabiliser $\mathbb{G}'_{k,p}$ in $SL(n)$, and the closure of its $SL(n)$ orbit

$$\overline{SL(n)(p_{k,p} \otimes (e_1 \wedge \dots \wedge e_p)^{\otimes K})}$$

is the union of the orbit of $p_{k,p} \otimes (e_1 \wedge \dots \wedge e_p)^{\otimes K}$ and finitely many other $SL(n)$ -orbits, all of which have codimension at least two if k is large enough (depending on p) and M is sufficiently large (depending on k and p).

The proof of Conjecture 9.1 should be similar to that of Theorem 6.2, with the rôle of the Borel subgroup B_k of $\mathrm{SL}(k)$ played by the standard parabolic subgroup $P \subset \mathrm{SL}(n)$ which stabilises the filtration

$$0 \subset \mathbb{C}^p = \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_p \subset \mathbb{C}^p \oplus \mathrm{Sym}^2 \mathbb{C}^p \subset \dots \subset \mathbb{C}^p \oplus \mathrm{Sym}^2 \mathbb{C}^p \oplus \dots \oplus \mathrm{Sym}^k \mathbb{C}^p = \mathbb{C}^n.$$

It follows immediately from Conjecture 9.1 that we would have

Conjecture 9.2. *If $p \geq 1$ and k is large enough (depending on p) then the reparametrisation group $\mathbb{G}'_{k,p}$ is a subgroup of the special linear group $\mathrm{SL}(\mathrm{sym}^{\leq k} p)$, where*

$$\mathrm{sym}^{\leq k} p = \sum_{i=1}^k \dim \mathrm{Sym}^i \mathbb{C}^p = \binom{k+p-1}{k-1},$$

such that the algebra of invariants

$$\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}}$$

is finitely generated, so that every linear action of $\mathbb{G}_{k,p}$ or $\mathbb{G}'_{k,p}$ on an affine or projective variety (with respect to an ample linearisation) which extends to a linear action of $\mathrm{GL}(\mathrm{sym}^{\leq k} p)$ has finitely generated invariants.

In particular we would have

Conjecture 9.3. *If $p \geq 1$ and k is large enough (depending on p) then the fibres $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ of the bundle $E_{k,p}^n$ are finitely generated graded complex algebras.*

We would also obtain geometric descriptions of the associated affine varieties

$$\mathrm{Spec}(\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}})$$

and $\mathrm{Spec}(\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}})$ generalising those in §7.

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